# INTERSECTION COHOMOLOGY ON NONRATIONAL POLYTOPES

PAUL BRESSLER AND VALERY A. LUNTS

## 1. Introduction

For an $n$-dimensional convex polytope $Q$ R. Stanley ([S]) defined a set of integers $h(Q) = (h_0(Q), h_1(Q), \ldots, h_n(Q))$ - the "generalized $h$-vector" - which are supposed to be the intersection cohomology Betti numbers of the toric variety $X_Q$ corresponding to $Q$. In case $Q \subset \mathbb{R}^n$ is a rational polytope the variety $X_Q$ indeed exists, and it is known ([S]) that $h_i(Q) = \dim IH^{2i}(X_Q)$. Thus, for a *rational polytope* $Q$, the integers $h_i(Q)$ satisfy

1. $h_i(Q) \geq 0$,
2. $h_i(Q) = h_{n-i}(Q)$ (Poincaré duality),
3. $h_0(Q) \leq h_1(Q) \leq \ldots \leq h_{[n/2]}(Q)$ (follows from the Hard Lefschetz theorem for projective algebraic varieties).

For an arbitrary convex polytope (more generally for an Eulerian poset) Stanley proved ([S],thm 2.4) the property 2 above. He conjectured that 1 and 3 also hold without the rationality hypothesis. This is still not known.

In this paper we propose an approach which we expect to lead to a proof of 1 and 3 for general convex polytopes. Our approach is modeled on the "equivariant geometry" of the (non-existent) toric variety $X_Q$ as developed in [BL].

Namely, given a convex polytope $Q \subset \mathbb{R}^n$ we consider the corresponding complete fan $\Phi = \Phi_Q$ in $\mathbb{R}^n$ and work with $\Phi$ instead of $Q$. Let $A$ denote the graded ring of polynomial functions on $\mathbb{R}^n$. Viewing $\Phi$ as a partially ordered set (of cones) we consider a category of sheaves of $A$-modules on $\Phi$. In this category we define a *minimal* sheaf $\mathcal{L}_\Phi$ which corresponds to the $T$-equivariant intersection cohomology complex on $X_Q$ if the latter exists. Our first main result is the "elementary" decomposition theorem for the direct image of the minimal sheaf under subdivision of fans (Theorem 5.5). (Recall, that a subdivision of a fan corresponds to a proper morphism of toric varieties.) We also develop the Borel-Moore-Verdier duality in the derived category of sheaves of

This research was supported in part by the NSF.

$A$-modules on $\Phi$. We show that $\mathcal{L}_\Phi$ is isomorphic to its Verdier dual (Corollary 6.26).

*Remark* 1.1. In fact the usual (equivariant) decomposition theorem for a proper morphism of toric varieties can be deduced from this "elementary" one by the equivalence of categories proved in [L] (Thm 2.6). However the proof of this last result by itself uses the fundamental properties of the intersection cohomology.

The minimal sheaf $\mathcal{L}_\Phi$ gives rise in a natural way to the graded vector space $IH(\Phi)$ which we declare to be the *intersection cohomology* of $\Phi$. (For rational $Q$ it is proved in [BL] that there is an isomorphism $IH(\Phi_Q) \cong IH(X_Q)$.) Let $ih_i(\Phi) = \dim IH^i(\Phi)$. We establish the following properties of $IH(\Phi)$:

1. $\dim IH(\Phi) < \infty$;
2. $ih_i(\Phi) = 0$, unless $i$ is even and $0 \leq i \leq 2n$;
3. $ih_0(\Phi) = ih_{2n}(\Phi) = 1$;
4. $ih_{n-i}(\Phi) = ih_{n+i}(\Phi)$.

Moreover, there is a natural operator $l$ of degree 2 on the space $IH(\Phi)$, which we expect to have the Lefschetz property as conjectured below:

**Conjecture 1.2.** *For each $i \geq 1$ the map*

$$l^i : IH^{n-i}(\Phi) \longrightarrow IH^{n+i}(\Phi)$$

*is an isomorphism.*

So far we were unable to prove this conjecture, but it seems to be within reach. In case $Q$ is rational the conjecture follows from the results in [BL]. This conjecture has the following standard corollary:

**Corollary 1.3** (of the conjecture)**.** *$ih_i(Q) \leq ih_{i+2}(Q)$ for $0 \leq i < n$.*

In fact the above conjecture implies "everything":

**Corollary 1.4.** *Assume the above conjecture is true. Then*

1. *$IH(\Phi_Q)$ is a combinatorial invariant of $Q$, i.e. it depends only on the face lattice of $Q$.*
2. *Moreover, $ih_{2j}(\Phi_Q) = h_j(Q)$ hence the $h$-vector $h(Q)$ has the properties conjectured by R. Stanley.*

The paper is organized as follows.

- Section 2 gives a brief account of our methods and main results.
- Section 3 discusses the elementary properties of the category of abelian sheaves on a fan and their cohomology.

- In Section 4 we endow a fan with the structure of a ringed space, single out a category of sheaves of modules over the structure sheaf and obtain the first "geometric" result (Theorem 4.7).
- In Section 5 we prove that our category of sheaves is semi-simple and identify the simple objects (Theorem 5.3). We show that our categories of sheaves are stable by direct image under morphisms induced by subdivision of fans (Theorem 5.5).
- Section 6 contains an account of duality on our category of sheaves. As a consequence we obtain the Poincare duality for $IH(\Phi)$.
- In Section 7 we make precise Lefschetz type conjectures and discuss their consequences.
- In Section 8 we apply the machinery to a conjecture of G. Kalai (proven recently in the rational case by T. Braden and R.D. MacPherson) and give our version of the proof.

## 2. Summary of methods and results

1. **Fans as ringed spaces.** Our point of departure is the observation that a fan $\Phi$ in a (real) vector space $V$ gives rise to a topological space, which we will denote by $\Phi$ as well, and a sheaf of graded rings $\mathcal{A}_\Phi$ on it. Namely, the points of $\Phi$ are cones, open subsets are subfans, and the stalk $\mathcal{A}_{\Phi,\sigma}$ of $\mathcal{A}_\Phi$ at the cone $\sigma \in \Phi$ is the graded algebra of polynomial functions on $\sigma$ (equivalently on the linear span of $\sigma$) and the structure maps are given by restriction of functions. All of these rings are quotients of the graded algebra $A = A_V$ of polynomial functions on $V$. The grading is assigned so that the linear functions have degree two.

In case $V$ is the Lie algebra of a torus $T$ the graded ring $A$ is canonically isomorphic to $H^*(BT)$ – the cohomology ring of the classifying space of $T$.

In the case of a rational fan $\Phi$ one has the (unique) normal $T$-toric variety $X_\Phi$ such that the $T$-orbits in $X_\Phi$ are in bijective correspondence with the cones of $\Phi$ and $\mathcal{A}_{\Phi,\sigma}$ is canonically isomorphic to the cohomology ring of the classifying space of the stabilizer of the corresponding orbit.

All $\mathcal{A}_\Phi$-modules will be regarded by default as $A$-modules. Let $A^+$ denote the ideal of functions which vanish at the origin. For a graded $A$-module $M$ we will denote by $\overline{M}$ the graded vector space $M/A^+M$.

2. **A category of $\mathcal{A}_\Phi$-modules.** To each fan $\Phi$ viewed as the ringed space $(\Phi, \mathcal{A}_\Phi)$ we associate the additive category $\mathfrak{M}(\mathcal{A}_\Phi)$ of (sheaves of finitely generated, graded) $\mathcal{A}_\Phi$-modules which are flabby and locally free over $\mathcal{A}_\Phi$. This latter condition means that, for an object $\mathcal{M}$ of

$\mathfrak{M}(\mathcal{A}_\Phi)$, the stalk $\mathcal{M}_\sigma$ is a free graded module of finite rank over $\mathcal{A}_{\Phi,\sigma}$. The flabbyness condition may be restated as follows: for every cone $\sigma$ the restriction map $\mathcal{M}_\sigma \to \mathcal{M}(\partial\sigma)$ is surjective (where $\mathcal{M}(\partial\sigma)$ is the space of section of $\mathcal{M}$ over the subfan $\partial\sigma$ consisting of cones properly contained in $\sigma$). It is easy to see that the sheaf $\mathcal{A}_\Phi$ is flabby if and only if the fan $\Phi$ is simplicial.

In the rational case the category $\mathfrak{M}(\mathcal{A}_\Phi)$ is equivalent to the category of equivariant perverse (maybe shifted) sheaves on $X_\Phi$. The following theorems verify that the category $\mathfrak{M}(\mathcal{A}_\Phi)$ and the cohomology of an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$ have the expected properties.

Since, by definition, the objects of $\mathfrak{M}(\mathcal{A}_\Phi)$ are flabby sheaves, it follows that, for $\mathcal{M}$ in $\mathfrak{M}(\mathcal{A}_\Phi)$, $H^i(\Phi;\mathcal{M}) = 0$ for $i \neq 0$.

**Theorem 2.1.** *Suppose that $\Phi$ is complete (i.e. the union of the cones of $\Phi$ is all of $V$), and $\mathcal{M}$ is in $\mathfrak{M}(\mathcal{A}_\Phi)$. Then, $H^0(\Phi;\mathcal{M})$ is a free $A$-module.*

In the rational case, $H^0(\Phi;\mathcal{M})$ is the equivariant cohomology of the corresponding perverse sheaf on $X_\Phi$.

The proof of Theorem 2.1 rests on the observation that the cohomology of a sheaf $\mathcal{F}$ on a complete fan may be calculated by a "cellular" complex $C^\bullet(\mathcal{F})$ whose component in degree $i$ is the direct sum of the stalks of $\mathcal{F}$ at cones of codimension $i$ and the differential is given by the sum (with suitable signs) of the restriction maps. In particular, if the sheaf $\mathcal{F}$ is flabby, then the complex $C^\bullet(\mathcal{F})$ is acyclic except in degree zero. This proves the conjecture of J. Bernstein and the second author (Conjecture 15.9 of [BL]) on the acyclicty properties of the "minimal complex", which happens to be the "cellular complex" of the simple object $\mathcal{L}_\Phi$ (see below) of $\mathfrak{M}(\mathcal{A}_\Phi)$. In the simplicial case (when $\mathcal{L}_\Phi \overset{\sim}{=} \mathcal{A}_\Phi$) an "elementary" proof of this fact was given by M. Brion in [B].

Concerning the structure of the category $\mathfrak{M}(\mathcal{A}_\Phi)$ we have the following result.

**Theorem 2.2.** *Every object in $\mathfrak{M}(\mathcal{A}_\Phi)$ is a finite direct sum of indecomposable ones. The indecomposable objects are, up to a shift of the grading, in bijective correspondence with the set of cones. (see Theorem 5.3 below).*

3. $IH$ **and** $IP$**.** The indecomposable object of $\mathfrak{M}(\mathcal{A}_\Phi)$ which corresponds to the cone $\sigma$ is a sheaf supported on the star of $\sigma$ (which constitutes the closure of the set $\{\sigma\}$ in our topology). Let $\mathcal{L}_\Phi$ denote the indecomposable object of $\mathfrak{M}(\mathcal{A}_\Phi)$ which is supported on all of $\Phi$ (the star of the origin of $V$) and whose stalk at the origin is the one

dimensional vector space in degree zero. The fan $\Phi$ is simplicial if and only if $\mathcal{L}_\Phi \overset{\sim}{=} \mathcal{A}_\Phi$.

In the rational case, when $\Phi$ is complete (and so is $X_\Phi$), the $A$-module $H^0(\Phi; \mathcal{L}_\Phi)$ is isomorphic to the $T$-equivariant intersection cohomology $IH_T(X_\Phi)$ of $X_\Phi$ and $\overline{IH_T(X_\Phi)}$ is the usual (non-equivariant) intersection cohomology of $X_\Phi$. This motivates the following notation:

**Definition 2.3.** Let $\Phi$ be a fan in $V$. Put

$$IH(\Phi) \overset{def}{=} \overline{H^0(\Phi; \mathcal{L}_\Phi)}$$

and denote by $ih(\Phi)$ the corresponding Poincaré polynomial.

For each cone $\sigma \in \Phi$ we may consider the corresponding local Poincaré polynomial. Namely, in the rational case the graded vector space $\overline{\mathcal{L}_{\Phi,\sigma}}$ is the (cohomology of the) stalk on the corresponding $T$-orbit $O_\sigma$ of the intersection cohomology complex of $X_\Phi$. A normal slice to $O_\sigma$ is an affine cone over some projective variety $Y_\sigma$. Then $\overline{\mathcal{L}_{\Phi,\sigma}}$ is the primitive part of the intersection cohomology of $Y_\sigma$. This motivates the following notation.

**Definition 2.4.** For $\sigma \in \Phi$ put

$$IP(\sigma) := \overline{\mathcal{L}_{\Phi,\sigma}}$$

and denote by $ip(\sigma)$ the corresponding Poincaré polynomial.

As is well known, the projectivity of a toric variety translates into the following picture. Suppose that $\Phi$ is a complete fan in $V$ and $l \in \mathcal{A}_\Phi(\Phi)$ is a (continuous) cone-wise linear (with respect to $\Phi$) strictly convex function on $V$. Multiplication by $l$ is an endomorphism (of degree 2) of $\mathcal{L}_\Phi$, $H^0(\Phi; \mathcal{L}_\Phi)$ and $IH(\Phi)$. In the rational case it is the Lefschetz operator on $IH(\Phi) = IH(X_\Phi)$ for the corresponding projective embedding of $X_\Phi$. Thus we make the following conjecture.

**Conjecture 2.5.** *(Hard Lefschetz) Let $\Phi$ be a complete fan. Multiplication by $l$ is a Lefshetz operator on $IH(\Phi)$ i.e. for each $i \geq 1$ the map*

$$l^i : IH^{n-i}(\Phi) \to IH^{n+i}(\Phi)$$

*is an isomorphism.*

4. **Subdivision and the decomposition theorem.** A fan $\Psi$ is a subdivision of a fan $\Phi$ if every cone of the latter is a union of cones of the former. In this case there is a morphism of ringed spaces $\pi : (\Psi, \mathcal{A}_\Psi) \longrightarrow (\Phi, \mathcal{A}_\Phi)$. In the rational case subdivision corresponds to a proper birational morphism of $T$-toric varieties.

**Theorem 2.6.** *(Decomposition Theorem) The functor of direct image under subdivision restricts to the functor* $\pi_* : \mathfrak{M}(\mathcal{A}_\Psi) \to \mathfrak{M}(\mathcal{A}_\Phi)$.

It should be pointed out that the only non-trivial part of Theorem 2.6 is the fact that the direct image of a locally free flabby sheaf is locally free which is proven by essentially the same argument as the one used in the proof of Theorem 2.1.

Combining Theorem 2.6 with Theorem 2.2 we obtain the statement which in the rational case amounts to the Decompostion Theorem of A. Beilinson J. Bernstein, P. Deligne, and O. Gabber ([BBD]) and its equivariant analog ([BL]) for proper birational morphisms of toric varieties: "the direct image of a pure object is a direct sum of (suitably shifted) pure objects". Continuing with notations introduced above we have the following "estimate":

**Corollary 2.7.** $\pi_*\mathcal{L}_\Psi$ *contains* $\mathcal{L}_\Phi$ *as a direct summand, therefore* $IH(\Psi)$ *contains* $IH(\Phi)$ *as a direct summand. So there is an inequality*

$$ih(\Psi) \geq ih(\Phi)$$

*(coefficient by coefficient) of polynomials with non-negative coefficients.*

5. **Duality.** As is well known, the (middle perversity) intersection cohomology of a compact space admits an intersection pairing (and the same is the case in the equivariant setting). To this end we have the following version of Borel-Moore-Verdier duality which we develop for the derived category of sheaves of $A$-modules on $\Phi$. One of the results is the following

**Theorem 2.8.** *Let* $\Phi$ *be a fan in* $V$. *There is a contravariant involution* $\mathbb{D}$ *on* $\mathfrak{M}(\mathcal{A}_\Phi)$ *(i.e. a functor* $\mathbb{D} : \mathfrak{M}(\mathcal{A}_\Phi)^{op} \to \mathfrak{M}(\mathcal{A}_\Phi)$ *and an isomorphism of functors* $\mathbb{D} \circ \mathbb{D} \stackrel{\sim}{=} \mathrm{Id}$*). If* $\Phi$ *is complete then there is a natural* $A$*-linear non-degenerate pairing*

$$H^0(\Phi; \mathcal{M}) \otimes_A H^0(\Phi; \mathbb{D}(\mathcal{M})) \to \omega_{A/\mathbb{R}}$$

*for every object* $\mathcal{M}$ *of* $\mathfrak{M}(\mathcal{A}_\Phi)$.

Here $\omega_{A/\mathbb{R}} = A \otimes \det V^*$ is the dualizing $A$-module, free of rank one, generated in degree $2\dim_{\mathbb{R}} V$ in accordance with our grading convention.

If follows from Theorem 2.8 that $\mathbb{D}$ is an anti-equivalence of categories, so that the dual of an indecomposable object is an indecomposable one. One checks immediately that the dual $\mathbb{D}(\mathcal{L}_\Phi)$ of $\mathcal{L}_\Phi$ has the properties which characterize the latter. Therefore there is a (non-canonical) isomorphism $\mathbb{D}(\mathcal{L}_\Phi) \stackrel{\sim}{=} \mathcal{L}_\Phi$. The numerical consequence of the auto-duality of $\mathcal{L}_\Phi$ is given below.

**Corollary 2.9.** *For a complete fan $\Phi$ in a vector space of dimension $n$ the polynomial $ih(\Phi)$ satisfies $ih_{n-k}(\Phi) = ih_{n+k}(\Phi)$.*

6. **Kalai type inequalities.** As an application of our technology we give our restatement of the inequality conjectured by G. Kalai, proven in the rational case by T. Braden and R.D. MacPherson in [BM]. Namely, suppose that $\Phi$ is a fan in $V$ generated by a single cone $\sigma$, i.e. $\Phi = [\sigma]$ and $\tau \subset \sigma$. Let $\mathrm{Star}(\tau)$ be the collection of cones in $\Phi$ which contain $\tau$. Consider the minimal sheaf $\mathcal{L}^{\tau}_{[\sigma]}$ on $[\sigma]$ which is based at $\tau$ (see 5.1 below) and put

$$IP(\mathrm{Star}(\tau)) := \overline{\mathcal{L}^{\tau}_{[\sigma],\sigma}}.$$

**Theorem 2.10.** *There is an inequality, coefficient by coefficient, of polynomials with non-negative coefficients*

$$ip(\sigma) \geq ip(\tau)ip(\mathrm{Star}(\tau)).$$

## 3. Abelian sheaves on fans

1. **Fans.** A *fan* $\Phi$ in a real vector space $V$ of dimension $\dim V = n$ is a collection of closed convex polyhedral cones with vertex at the origin $\underline{o}$ satisfying

- any two cones in $\Phi$ intersect along a common face;
- if a cone is in $\Phi$, then so are all of it's faces.

A fan has a structure of a partially ordered set: given cones $\sigma$ and $\tau$ in $\Phi$ we write $\tau \leq \sigma$ if $\tau$ is a face of $\sigma$.

The origin is the unique minimal cone in every fan and will be denoted $\underline{o}$.

Let $d(\sigma)$ denote the dimension of the cone $\sigma$. Thus $d(\sigma) = 0$ iff $\sigma = \underline{o}$.

The fan $\Phi$ is *complete* if and only if the union of cones of $\Phi$ is all of $V$. The fan $\Phi$ is *simplicial* if every cone of $\Phi$ is simplicial. A cone of dimension $k$ is *simplicial* if it has $k$ one-dimensional faces (rays).

2. **Topology on a fan.** The (partially ordered) $\Phi$ will be considered as a topological space with the open sets the subfans of $\Phi$. A subset, say $S$, of $\Phi$ generates a subfan, denoted $[S]$.

An open subset is *irreducible* if it is not a union of two open subsets properly contained in it. The irreducible open sets are the subfans generated by single cones. We will frequently abuse notation and write $[\sigma]$ for the irreducible open set $[\{\sigma\}]$.

Note that the topological space $\Phi$ has the following property: the intersection of irreducible open sets is irreducible.

Let $\Phi_{\leq k}$ denote the subset of cones of dimension at most $k$; this is an open subset of $\Phi$.

Let $\sigma$ be a cone in $\Phi$. Denote by $\operatorname{Star}(\sigma) \subset \Phi$ the subset of all the cones $\tau$ such that $\sigma \leq \tau$. This is a closed subset of $\Phi$. Its image under the projection $V \to V/\operatorname{Span}(\sigma)$ is a fan that will be denoted by $\overline{\operatorname{Star}(\sigma)}$.

3. **Sheaves on a fan.** Regarding $\Phi$ as a topological space with open sets the subfans of $\Phi$, we consider sheaves on $\Phi$. Let $\mathcal{I}(\Phi)$ denote the partially ordered set of irreducible open sets of $\Phi$ and inclusions thereof. This partially ordered set is isomorphic to $\Phi$.

A sheaf on $\Phi$ restricts to a presheaf (a contravariant functor) on $\mathcal{I}(\Phi)$ and this correspondence is an equivalence of categories. Since, for a sheaf $F$ and a cone $\sigma$, the stalk $F_\sigma$ is equal to the sections $\Gamma([\sigma]; F)$ of $F$ over the corresponding irreducible open set, the sheaf $F$ is uniquely determined by the assignment $\sigma \mapsto F_\sigma$ and the restriction maps $F_\sigma \longrightarrow F_\tau$ whenever $\tau \leq \sigma$.

As usual the support $\operatorname{Supp}(F)$ of a sheaf $F$ is the closure of the set of $\sigma$'s, such that $F_\sigma \neq 0$.

For a cone $\sigma$ let $\partial\sigma$ denote the subfan generated by the proper faces of $\sigma$. Let $F$ be a sheaf and consider the following condition:

for every cone $\sigma$ the canonical map
$$F_\sigma \longrightarrow \Gamma(\partial\sigma; F) \text{ is surjective.} \tag{3.1}$$

**Lemma 3.1.** *A sheaf $F$ satisfies the condition* (3.1) *if and only if it is flabby.*

*Proof.* The condition is obviously necessary. To see that it is sufficient we need to show that a section of $F$ defined over a subfan $\Psi$ extends to a global section. Clearly, it is sufficient to show that a section defined over $\Psi \bigcup \Phi_{\leq k}$ extends to a section defined over $\Psi \bigcup \Phi_{\leq k+1}$, but this is immediate from (3.1). □

For the rest of this section 3 we restrict our attention to the category of sheaves of $\mathbb{R}$-vector spaces on $\Phi$ which we will denote by $\operatorname{Sh}(\Phi)$.

4. **Some elementary properties of the category $\operatorname{Sh}(\Phi)$.** For a cone $\sigma$ let $i_\sigma : \{\sigma\} \hookrightarrow \Phi$ denote the inclusion. The embedding $i_\sigma$ is locally closed, and closed (respectively open) if and only if $\sigma$ is maximal (respectively minimal, i.e. the origin).

Suppose that $W$ is a vector space considered as a sheaf on $\{\sigma\}$. Then, clearly, $\mathbf{R}^p(i_\sigma)_*W = 0$ for $p \neq 0$, the sheaf $(i_\sigma)_*W$ is an injective object in $\operatorname{Sh}(\Phi)$ equal to the constant sheaf $W$ supported on $\operatorname{Star}(\sigma)$.

Let $i_{[\sigma]} : [\sigma] \hookrightarrow \Phi$ denote the open embedding of the irreducible open set $[\sigma]$. Then, the extension by zero $(i_{[\sigma]})_! W$ of the constant sheaf $W$ on $[\sigma]$ is a projective object in $\mathrm{Sh}(\Phi)$. Thus the abelian category $\mathrm{Sh}(\Phi)$ has enough projectives.

5. **The cellular complex of a sheaf.** Let $\Phi$ be a fan in $V \simeq \mathbb{R}^n$ and $F$ be a sheaf on $\Phi$. Choose an orientation of each cone in $\Phi$. The *cellular complex* $C^\bullet(F)$ of $F$ is defined as follows

$$C^\bullet(F) := 0 \to C^0(F) \to C^1(F) \to ... \to C^n(F) \to 0,$$

where

$$C^i(F) = \bigoplus_{d(\sigma)=n-i} F_\sigma$$

and the differential $d^i : C^i(F) \to C^{i+1}(F)$ is the sum of the restriction maps $F_\sigma \to F_\tau$ with the sign $\pm 1$ depending on whether the orientations of $\sigma$ and $\tau$ agree or disagree.

*Remark* 3.2. $C^\bullet(\cdot)$ is an exact functor from $\mathrm{Sh}(\Phi)$ to complexes of vector spaces.

**Proposition 3.3.** *Assume that the fan $\Phi$ is complete. Then the complex $C^\bullet(F)$ is quasi-isomorphic to $\mathbf{R}\,\Gamma(\Phi; F)$. In particular,*

$$H^i(C^\bullet(F)) = H^i(\Phi; F).$$

*Proof.* First notice that since $\Phi$ is complete we have

$$\Gamma(\Phi; F) = H^0(C^\bullet(F)).$$

Indeed, consider the open covering of $\Phi$ by maximal irreducible open sets $[\sigma]$, $d(\sigma) = n$. Then to give a global section $s \in \Gamma(\Phi; F)$ is the same as to give a collection of local sections $s_\sigma \in F([\sigma])$, $d(\sigma) = n$ such that $s_\sigma = s_\tau$ in $F([\sigma \cap \tau])$ if $d(\sigma \cap \tau) = n - 1$. This shows that $\Gamma(\Phi; F) = H^0(C^\bullet(F))$.

To prove the proposition consider an injective resolution

$$F \to I^0 \to I^1 \to ...,$$

where sheaves $I^j$ are direct sums of constant sheaves on closed subsets $\mathrm{Star}(\sigma)$, $\sigma \in \Phi$ extended by zero to $\Phi$. Since $C^\bullet(\cdot)$ is an exact functor it suffices to prove the following claim.

**Claim 3.4.** *Let $\Psi$ be a fan in $V$, $\sigma \in \Psi$ and $W$ – a vector space. Assume that the fan $\overline{\mathrm{Star}(\sigma)}$ in $V/\operatorname{Span}(\sigma)$ is complete. Let $I$ be the constant sheaf $W$ on $\mathrm{Star}(\sigma)$ extended by zero to $\Psi$. Then*

$$H^i(C^\bullet(I)) = \begin{cases} W & if \quad i = 0 \\ 0, & otherwise. \end{cases}$$

*Proof of claim.* Since $\underline{\mathrm{Star}(\sigma)}$ is a closed subset of $\Psi$ which is isomorphic to the complete fan $\overline{\mathrm{Star}(\sigma)}$ (of dimension $n - d(\sigma)$) we may assume that $\sigma = \underline{o}$ and hence $\mathrm{Star}(\sigma) = \Psi$ is a complete fan in $V$. Then $C^\bullet(I)$ is isomorphic to a cellular cochain complex of a ball of dimension $n$ (with coefficients $W$). This proves the claim and the proposition.

□

**Corollary 3.5.** *Let $\sigma \subset V$ be a cone and $\Psi = [\partial\sigma]$ be the fan generated by the boundary $\partial\sigma$. Then for any sheaf $F$ on $\Psi$ the shifted cellular complex $C^\bullet(F)[n - d(\sigma) + 1]$ is quasi-isomorphic to $\mathbf{R}\Gamma(\Psi; F)$.*

*Proof.* This follows from the above proposition since the fan $\Psi$ is isomorphic to a complete fan in $\mathbb{R}^{d(\sigma)-1}$. □

Later on we will need the following version of the previous proposition.

**Proposition 3.6.** *Let $\Phi$ be a fan in $V$, not necessarily complete. Let $F \in \mathrm{Sh}(\Phi)$ be such that its support $\underline{Z = Supp}(F)$ satisfies the following condition: for each $\sigma \in Z$ the fan $\overline{\mathrm{Star}(\sigma)}$ in $V/\mathrm{Span}(\sigma)$ is complete. Then the cellular complex $C^\bullet(F)$ is quasi-isomorphic to $\mathbf{R}\Gamma(\Phi; F)$.*

*Proof.* Same as that of Proposition 3.3. □

The next three lemmas will be used later on.

**Lemma 3.7.** *Let $\sigma \subset V$ be a cone of positive dimension, i.e. $\sigma \neq \underline{o}$, and $\Phi = [\sigma]$ be the fan generated by $\sigma$. Let $F$ be a constant sheaf on $\Phi$. Then the cellular complex $C^\bullet(F)$ is acyclic.*

*Proof.* We may assume that $d(\sigma) = dimV = n$. Then the cellular complex $C^\bullet(F)$ is isomorphic to an augmented chain complex of a ball of dimension $n - 1$. This proves the lemma. □

*Remark* 3.8. In the notation of the previous lemma notice that if $F \neq 0$, then $H^0([\sigma]; F) \neq 0$. Thus for a fan which is not complete the cellular complex does not necessarily compute the cohomology of the sheaf.

**Lemma 3.9.** *Let $\sigma$ be a cone in $V$ and consider the fan $[\sigma]$. Let $F$ be a flabby sheaf on the fan $[\sigma]$. Then the cellular complex $C^\bullet(F)$ is acyclic except in the lowest degree, i.e. $H^i(C^\bullet(F)) = 0$ for $i \neq n - d(\sigma)$.*

*Proof.* Put $\Psi := [\partial\sigma]$ and denote $F_\Psi := F|_\Psi$. Then by Corollary 3.5 $H^{i+(n-d(\sigma)+1)}(C^\bullet(F_\Psi)) = H^i(\Psi; F)$. Since $F_\Psi$ is flabby $H^j(C^\bullet(F_\Psi)) = 0$ if $j \neq n - d(\sigma) + 1$, and $H^{n-d(\sigma)+1}(C^\bullet(F_\Psi)) = \Gamma(\Psi; F)$. Again by the flabbiness of $F$ the map $F_\sigma \to \Gamma(\Psi; F)$ is surjective, hence $H^i(C^\bullet(F)) = 0$ for $i \neq n - d(\sigma)$. □

6. **Cohomology of some simple sheaves.**

**Lemma 3.10.** *Let $\sigma \subset V$ be a cone and consider the fan $\Phi = [\sigma]$. Then for any sheaf $F$ on $\Phi$ $H^i(\Phi, F) = 0$, if $i > 0$.*

*Proof.* Let $\mathbb{R}_\Phi$ be the constant sheaf on $\Phi$ with stalk $\mathbb{R}$. We have the isomorphism of functors

$$\Gamma(\Phi, \cdot) = \mathrm{Hom}(\mathbb{R}_\Phi, \cdot).$$

But $\mathbb{R}_\Phi$ is a projective object in $Sh(\Phi)$. This proves the lemma. □

**Lemma 3.11.** *Let $\Phi$ be a fan in $V$ and $F \in Sh(\Phi)$ be the constant sheaf on $\Phi$ with stalk $W$. Then*

$$H^i(\Phi, F) = \begin{cases} W, & if \quad i = 0 \\ 0, & otherwise. \end{cases}$$

*Proof.* Since the space $\Phi$ is connected $H^0(\Phi, F) = W$. The rest follows from the injectivity of the sheaf $F$. □

**Lemma 3.12.** *Let $\Phi$ be a complete fan in $V$ and $W$ be a vector space. Consider the sheaf $W_{\underline{o}}$ on $\Phi$ which is the extension by zero of the sheaf $W$ on the open point $\underline{o}$. Then*

$$H^i(\Phi, W_{\underline{o}}) = \begin{cases} W, & if \quad i = n \\ 0, & otherwise. \end{cases}$$

*Proof.* Put $Z := \Phi - \{\underline{o}\}$ and let $i : Z \hookrightarrow \Phi$ be the corresponding closed embedding. Let $W_\Phi$, $W_Z$ denote the constant sheaves with stalk $W$ on $\Phi$ and $Z$ respectively. We have the exact sequence

$$0 \to W_{\underline{o}} \to W_\Phi \to i_* W_Z \to 0.$$

By Lemma 3.11 $H^0(\Phi, W_\Phi) = W$ and $H^i(\Phi, W_\Phi) = 0$ if $i > 0$. On the other hand the cellular complex $C^\bullet(i_* W_Z)$ is isomorphic to a chain complex of a sphere of dimension $n-1$. Hence by Proposition 3.6

$$H^j(\Phi, i_* W_Z) = \begin{cases} W, & if \quad j = 0, n-1 \\ 0, & otherwise \end{cases}$$

if $n > 1$ and

$$H^j(\Phi, i_* W_Z) = \begin{cases} W \oplus W, & if \quad j = 0 \\ 0, & otherwise \end{cases}$$

if $n = 1$. The map $H^0(\Phi, W_\Phi) \to H^0(\Phi, i_* W_Z)$ is injective, so the lemma follows. □

7. **The cocellular complex.** Let $\Phi$ be a fan in $V$ and $\mathbb{R}_{\underline{o}}$ be the extension by zero to $\Phi$ of the sheaf $\mathbb{R}$ on the open point $\underline{o} \in \Phi$. We want to construct an injective resolution of $\mathbb{R}_{\underline{o}}$ of a special kind. For a cone $\sigma \in \Phi$ let $i_\sigma : \{\sigma\} \hookrightarrow \Phi$ denote the inclusion. Consider the sheaf $i_{\sigma *}\mathbb{R}$ on $\Phi$. This is a constant sheaf on $\mathrm{Star}(\sigma)$ extended by zero to $\Phi$. Such sheaves are injective. Note that for $\tau \leq \sigma$ we have the natural surjective morphism $r_{\tau\sigma} : i_{\sigma *}\mathbb{R} \to i_{\tau *}\mathbb{R}$. Consider the complex

$$K^\bullet = K^\bullet_\Phi := K^{-n} \xrightarrow{d^{-n}} K^{-n+1} \xrightarrow{d^{-n+1}} \ldots \xrightarrow{d^{-1}} K^0,$$

where

$$K^{-n+j} = \bigoplus_{d(\sigma)=j} i_{\sigma *}\mathbb{R},$$

and the differential $d$ is the sum of maps $r_{\tau\sigma}$ with $\pm$ sign depending on the compatibility of orientations of $\sigma$ and $\tau$.

Note that $K^{-n}$ is the constant sheaf $\mathbb{R}$ on $\Phi$ and $\operatorname{Ker} d^{-n} = \mathbb{R}_{\underline{o}}$.

**Lemma 3.13.** *The complex $K^\bullet$ is an injective resolution of the sheaf $\mathbb{R}_{\underline{o}}[n]$.*

*Proof.* The injectivity of $K^\bullet$ was noted before. It remains to show that for $\underline{o} \neq \sigma \in \Phi$ the complex of stalks

$$K^{-n}_\sigma \to K^{-n+1}_\sigma \to \ldots \to K^0_\sigma$$

is exact. Since $i_{\tau *}\mathbb{R}$ is the constant sheaf on $\mathrm{Star}(\tau)$ the above complex is isomorphic to the dual of the cellular complex $C^\bullet(\mathbb{R}_{[\sigma]})$. This last complex is acyclic by Lemma 3.7 above.

∎

**Definition 3.14.** Given a sheaf $F$ of $\mathbb{R}$-vector spaces on $\Phi$ we define its *cocellular complex* $K^\bullet(F) = K^\bullet_\Phi(F)$ as

$$K^\bullet(F) := \mathrm{Hom}^\bullet(F, K^\bullet).$$

*Remark* 3.15. The cocellular complex has the following properties.

1. $K^i(F) = 0$ unless $i \in [-n, 0]$.
2. By the above lemma $K^\bullet(F) = \mathbf{R}\,\mathrm{Hom}^\bullet(F, \mathbb{R}_{\underline{o}}[n])$.
3. Note that $\mathrm{Hom}(F, i_{\sigma *}\mathbb{R}) = \mathrm{Hom}(F_\sigma, \mathbb{R})$. Thus the cocellular complex $K^\bullet(F)$ is just the dual of the cellular complex $C^\bullet(F)$:

$$K^\bullet(F) = \mathrm{Hom}^\bullet(C^\bullet(F), \mathbb{R}).$$

4. An appropriate version of the cocellular complex will play a role in our discussion of the Borel-Moore-Verdier duality for sheaves of $A_\Phi$-modules (see Section 6) on complete fans. Of course, similar duality exists for sheaves of vector spaces, but we will not need it.

## 4. Fans as ringed spaces

1. **The structure sheaf of a fan.** Let $\Phi$ be a fan in $V$. Let $V^*$ denote the constant sheaf $\sigma \mapsto V^*$ on $\Phi$. Let $\Omega^1_\Phi$ denote the subsheaf of $V^*$ given by

$$\Phi \ni \sigma \mapsto \Omega^1_{\Phi,\sigma} \stackrel{def}{=} \sigma^\perp \subseteq V^* ,$$

where $\sigma^\perp$ denotes the subspace of linear functions which vanish identically on $\sigma$.

Let $\mathcal{G}$ denote the sheaf determined by the assignment

$$\Phi \ni \sigma \mapsto \mathcal{G}_\sigma = \mathrm{Span}(\sigma)^* .$$

Thus, there is a short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_\Phi \longrightarrow V^* \xrightarrow{\pi} \mathcal{G} \longrightarrow 0 . \tag{4.1}$$

¿From now on $A$ will denote the symmetric algebra of $V^*$ with grading determined by assigning degree 2 to $V^*$. We will use the notation $A_\Phi$ for the corresponding constant sheaf on $\Phi$.

**Definition 4.1.** The structure sheaf $\mathcal{A}_\Phi$ is the symmetric algebra of $\mathcal{G}$, i.e. the sheaf of cone-wise polynomial functions, graded so that the linear functions have degree 2.

*Remark* 4.2. Clearly, there is an epimorphism of sheaves of graded algebras $A_\Phi \longrightarrow \mathcal{A}_\Phi$.

*Remark* 4.3. With these definitions $(\Phi, \mathcal{A}_\Phi)$ is a ringed space over the one point ringed space $(\emptyset, A)$ which we imagine as "the empty fan" in $V$.

In what follows "an $\mathcal{A}_\Phi$-module" will mean "a (locally) finitely generated graded $\mathcal{A}_\Phi$-module" and similarly for $A_\Phi$-modules. An $\mathcal{A}_\Phi$-module $\mathcal{M}$ is *locally free* if, for every cone $\sigma$, $\mathcal{M}_\sigma$ is a free (graded) $\mathcal{A}_{\Phi,\sigma}$-module.

Let $A^+$ denote the ideal of elements of positive degree. For an $A$-module $M$ we will denote by $\overline{M}$ the graded vector space $M/MA^+$.

For a graded $A$-module (or sheaf) $M = \oplus M_k$ denote by $M(t)$ the corresponding shifted object

$$M(t)_k = M_{k+t}.$$

The flabbiness criterion (3.1) applied to an $\mathcal{A}_\Phi$-module together with Nakayama's Lemma amounts to the following.

**Lemma 4.4.** *An $\mathcal{A}_\Phi$-module $\mathcal{M}$ is flabby if and only if for every cone $\sigma$ the canonical map*

$$\overline{\mathcal{M}_\sigma} \longrightarrow \overline{\Gamma(\partial\sigma; \mathcal{M})}$$

*is surjective.*

**Definition 4.5.** Let $\mathfrak{M}(\mathcal{A}_\Phi)$ denote the additive category of flabby locally free $\mathcal{A}_\Phi$-modules considered as a full subcategory of sheaves of $\mathcal{A}_\Phi$-modules and morphisms of dergee zero.

**Lemma 4.6.** *The fan $\Phi$ is* simplicial *if and only if the structure sheaf $\mathcal{A}_\Phi$ is flabby.*

*Proof.* The statement is easily seen to be equivalent to the following one: *Suppose that $P_1, \dots, P_n$ are polynomials in variables $x_1, \dots, x_n$ which satisfy $\dfrac{\partial P_i}{\partial x_i} = 0$ and $P_i|_{x_j=0} = P_j|_{x_i=0}$ for all $i$ and $j$. Then there is a polynomial $Q$ such that $Q|_{x_i=0} = P_i$.* Verification of the latter fact is left to the reader. □

2. **Cohomology of objects in $\mathfrak{M}(\mathcal{A}_\Phi)$ on complete fans.**

**Theorem 4.7.** *Suppose that $\Phi$ is a complete fan in $V$ and $\mathcal{M}$ is in $\mathfrak{M}(\mathcal{A}_\Phi)$. Then, $H^i(\Phi; \mathcal{M}) = 0$ for $i \neq 0$ and $H^0(\Phi; \mathcal{M})$ is a free $A$-module.*

*Proof.* The vanishing of higher cohomology is a direct consequence of the flabbiness of the objects of $\mathfrak{M}(\mathcal{A}_\Phi)$.

A graded $A$-module $M$ is free if and only if $\mathrm{Ext}^i_A(M, A) = 0$ for $i \neq 0$. By Proposition 3.3 above $H^0(\Phi; \mathcal{M}) = H^0 C^\bullet(\mathcal{M})$. Note that, for every $j$, $\mathrm{Ext}^i_A(C^j(\mathcal{M}), A) = 0$ for $i \neq j$. Since, in addition, $H^i C^\bullet(\mathcal{M}) = 0$ for $i \neq 0$ it follows by the standard argument that $\mathrm{Ext}^i_A(H^0 C^\bullet(\mathcal{M}), A) = 0$ for $i \neq 0$. □

## 5. Minimal sheaves and intersection cohomology

1. **Minimal sheaves.** Recall that, for a cone $\sigma$, the set $\mathrm{Star}(\sigma)$ is defined as the collection of those cones $\tau$ which satisfy $\tau \geq \sigma$. Namely, $\mathrm{Star}(\sigma)$ is the closure of the set $\{\sigma\}$.

Fix a cone $\sigma$ and consider the following conditions on an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$:

1. $\mathcal{M}_\sigma \neq 0$ and $\mathcal{M}_\tau \neq 0$ only if $\tau \in \mathrm{Star}(\sigma)$;
2. for every cone $\tau \in \mathrm{Star}(\sigma)$, $\tau \neq \sigma$, the canonical map $\overline{\mathcal{M}_\tau} \to \overline{\Gamma(\partial\tau; \mathcal{M})}$ is an isomorphism.

**Definition 5.1.** In what follows we will refer to an object as above as *a minimal sheaf based at $\sigma$*. An instance of a minimal sheaf $\mathcal{M}$ based at $\sigma$ with $\mathcal{M}_\sigma \cong \mathcal{A}_{\Phi,\sigma}$ will be denoted $\mathcal{L}^\sigma_\Phi$. We will also denote $\mathcal{L}_\Phi = \mathcal{L}^o_\Phi$

**Proposition 5.2.** *Let $\Phi$ be any fan.*

1. *For every $\sigma \in \Phi$ and every finitely generated graded $\mathcal{A}_{\Phi,\sigma}$-module $M$ there exists a unique (up to an isomorphism) minimal sheaf $\mathcal{M}$ based at $\sigma$ such that $\mathcal{M}_\sigma = M$. In particular, the minimal sheaf $\mathcal{L}_\Phi^\sigma$ exists for each $\sigma \in \Phi$.*
2. *Moreover, if $\mathcal{M}$ is a minimal sheaf based at $\sigma$ then*
$$\mathcal{M} \simeq \mathcal{L}_\Phi^\sigma \otimes_{\mathbb{R}} \overline{\mathcal{M}_\sigma}.$$
*In particular $\mathcal{M}$ is a direct sum of sheaves $\mathcal{L}_\Phi^\sigma(t)$, $t \in \mathbb{Z}$.*
3. *The minimal sheaves $\mathcal{L}_\Phi^\sigma(t)$, $t \in \mathbb{Z}$ are indecomposable objects in the category of $\mathcal{A}_\Phi$-modules.*
4. *Let $U \subset \Phi$ be an open subset, i.e. $U$ is a subfan. Then*
$$\mathcal{L}_\Phi|_U = \mathcal{L}_U.$$

*Proof.* Easy exercise. □

**Theorem 5.3.** *Let $\Phi$ be a fan in $V$. Every object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$ is isomorphic to a direct sum of minimal sheaves. In particular, $\mathcal{M}$ is a direct sum of indecomposable objects $\mathcal{L}_\Phi^\sigma(t)$, $t \in \mathbb{Z}$.*

*Proof.* The last part of the theorem follows from the first one using parts 2 and 3 of Proposition 5.2.

Consider an object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$. We will show that it is isomorphic to a direct sum of minimal sheaves by induction on
$$|\mathcal{M}| = \sum_\sigma rank_{\mathcal{A}_{\Phi,\sigma}} \mathcal{M}_\sigma.$$

Consider a cone $\sigma$ such that $\mathcal{M}_\sigma \neq 0$ and, for every $\tau \in \partial\sigma$, $\mathcal{M}_\tau = 0$. We will show that, for each such $\sigma$, $\mathcal{M}$ contains as a direct summand a minimal sheaf $\mathcal{K}$ based at $\sigma$ with $\mathcal{K}_\sigma = \mathcal{M}_\sigma$. That is, we will construct a direct sum decompostion
$$\mathcal{M} = \mathcal{K} \oplus \mathcal{N} \tag{5.1}$$
with $\mathcal{K}$ as above and $\mathcal{N}$ in $\mathfrak{M}(\mathcal{A}_\Phi)$. This is sufficient, since, clearly, $|\mathcal{N}| < |\mathcal{M}|$.

Note, that we need to specify the direct sum decompostion (5.1) only on Star$(\sigma)$. Therefore it is sufficient to treat the case when $\sigma$ is the origin $\underline{o}$.

We proceed to construct the decomposition (5.1) by induction on the dimension of the cone and the number of cones of the given dimension.

Let $\mathcal{K}_{\underline{o}} = \mathcal{M}_{\underline{o}}$ and $\mathcal{N}_{\underline{o}} = 0$. Assume that a direct sum decomposition
$$\mathcal{M}|_{\Phi_{\leq k}} = \mathcal{K}_{\leq k} \oplus \mathcal{N}_{\leq k}$$
in $\mathfrak{M}(\mathcal{A}_\Phi|_{\Phi_{\leq k}})$ has been defined and consider a cone $\sigma$ of dimension $k+1$. Since $\partial\sigma$ consists of cones of dimension at most $k$ the induction

hypothesis says that there is a direct sum decompostion of $\Gamma(\partial\sigma;\mathcal{A}_\Phi)$-modules
$$\Gamma(\partial\sigma;\mathcal{M}) = \Gamma(\partial\sigma;\mathcal{K}_{\leq k}) \oplus \Gamma(\partial\sigma;\mathcal{N}_{\leq k})\ .$$

**Claim 5.4.** *There is a decomposition*
$$\mathcal{M}_\sigma = K_\sigma \oplus N_\sigma$$
*into a direct sum of free $\mathcal{A}_{\Phi,\sigma}$-modules, such that the restriction homomorphism*
$$\mathcal{M}_\sigma \twoheadrightarrow \Gamma(\partial\sigma;\mathcal{M})$$
*maps $K_\sigma$ to $\Gamma(\partial\sigma;\mathcal{K}_{\leq k})$ and $N_\sigma$ to $\Gamma(\partial\sigma;\mathcal{N}_{\leq k})$ and induces an isomorphism*
$$\overline{K_\sigma} \xrightarrow{\sim} \overline{\Gamma(\partial\sigma;\mathcal{K}_{\leq k})}$$
*and an epimorphism*
$$\overline{N_\sigma} \twoheadrightarrow \overline{\Gamma(\partial\sigma;\mathcal{N}_{\leq k})}$$

Assume the claim for the moment. The desired extension $\mathcal{K}_{\leq k+1}$ (respectively $\mathcal{N}_{\leq k+1}$) of $\mathcal{K}_{\leq k}$ (respectively $\mathcal{N}_{\leq k}$) is given, for every cone $\sigma$ of dimension $k+1$, by $\mathcal{K}_{\leq k+1,\sigma} = K_\sigma$ (respectively $\mathcal{N}_{\leq k+1,\sigma} = N_\sigma$) and has all the required properties.

*Proof of claim.* Choose a subspace $Z \subset \Gamma(\partial\sigma;\mathcal{K}_{\leq k})$ which maps isomorphically onto $\overline{\Gamma(\partial\sigma;\mathcal{K}_{\leq k})}$ under the residue map. Choose a subspace $S \subset \mathcal{M}_\sigma$ so that the map $\mathcal{M}_\sigma \to \Gamma(\partial\sigma;\mathcal{M})$ restricts to an isomorphisms $S \xrightarrow{\sim} Z$. Since $S \bigcap A^+\mathcal{M}_\sigma = 0$ there is a subspace $T \subset \mathcal{M}_\sigma$ such that $S \bigcap T = 0$ and $S \oplus T$ generates $\mathcal{M}_\sigma$ freely. Subtracting, if necessary, elements of $A \cdot S$ from elements of $T$ we may assume that the image of $T$ under the map $\mathcal{M}_\sigma \to \Gamma(\partial\sigma;\mathcal{M})$ is contained in $\Gamma(\partial\sigma;\mathcal{N}_{\leq k})$. Thus we may take $K_\sigma = \mathcal{A}_{\Phi,\sigma}S$ and $N_\sigma = \mathcal{A}_{\Phi,\sigma}T$. This concludes the proof of the claim and of the theorem. □

2. **Subdivision of fans and the decomposition theorem.** Suppose that $\Phi$ and $\Psi$ are two fans in $V$ and $\Psi$ is a subdivision of $\Phi$, which is to say, every cone of $\Phi$ is a union of cones of $\Psi$. (In the rational case this induces a proper morphism of toric varieties). This corresponds to a morphism of ringed spaces $\pi : (\Psi,\mathcal{A}_\Psi) \to (\Phi,\mathcal{A}_\Phi)$. The next theorem combined with the structure Theorem 5.3 is a combinatorial analog of the decomposition theorem ([BBD],[BL]).

**Theorem 5.5.** *In the notations introduced above, for $\mathcal{M}$ in $\mathfrak{M}(\mathcal{A}_\Psi)$,*

1. $\mathbf{R}^i\pi_*\mathcal{M} = 0$ *for $i \neq 0$ and $\pi_*\mathcal{M}$ is flabby;*
2. $\pi_*\mathcal{M}$ *is locally free.*

*In other words, the direct image under subdivision $\pi$ restricts to an exact functor $\pi_* : \mathfrak{M}(\mathcal{A}_\Psi) \to \mathfrak{M}(\mathcal{A}_\Phi)$.*

*Proof.* The first claim follows from the flabbiness of $\mathcal{M}$.

Since the issue is local on $\Phi$ we may assume that the latter is generated by a single cone $\sigma$ of top dimension $n$, i.e. $\Phi = [\sigma]$. By induction on dimension it is sufficient to show that the stalk

$$(\pi_*\mathcal{M})_\sigma = H^0(\Phi; \pi_*\mathcal{M}) = H^0(\Psi; \mathcal{M})$$

is a free $A$-module.

Let $Z = \pi^{-1}(\sigma)$. This is a closed subset of $\Psi$ which consists of the cones which subdivide the interior of $\sigma$.

**Claim 5.6.** *For any sheaf $F$ on $\Psi$ the restriction map $H^0(\Psi; F) \to H^0(Z; F)$ is an isomorphism.*

*Proof of claim.* Indeed, a global section $\alpha \in \Gamma(\Psi; F)$ is the same as a collection of local sections $\alpha_\tau \in F_\tau = \Gamma([\tau]; F)$, $d(\tau) = n$ such that $\alpha_\tau = \alpha_\xi$ in $F_{\tau\cap\xi}$ in case $d(\tau \cap \xi) = n - 1$. The same local data specifies an element in $\Gamma(Z; F)$. This proves the claim.

Denote by $i : Z \hookrightarrow \Psi$ the corresponding closed embedding. Put

$$\mathcal{M}_Z := i_* i^* \mathcal{M}.$$

Then by the above claim $(\pi_*\mathcal{M})_\sigma = H^0(\Psi; \mathcal{M}_Z)$.

By Proposition 3.6 the cellular complex $C^\bullet(\mathcal{M}_Z)$ is quasi-isomorphic to $\mathbf{R}\Gamma(\Psi; \mathcal{M}_Z)$. Note that the sheaf $\mathcal{M}_Z$ is flabby. Now we show that the $A$-module $H^0(\Psi; \mathcal{M}_Z)$ is free by the same argument as in the proof of Theorem 4.7. □

3. **Intersection cohomology of fans.**

**Definition 5.7.** Let $\Phi$ be a fan in $V$. Define its *intersection cohomology* as the graded vector space

$$IH(\Phi) := \overline{H^0(\Phi; \mathcal{L}_\Phi)}$$

and denote by $ih(\Phi)$ the corresponding Poincare polynomial.

**Lemma 5.8.** *1.* $\dim IH(\Phi) < \infty$.

*2.* $ih_j(\Phi) = 0$ *for* $j < 0$ *or* $j$ *odd.*

*3.* $ih_0(\Phi) = 1$.

*Proof.* The $A$-module $H^0(\Phi; \mathcal{L}_\Phi)$ is finitely generated, so 1) follows. The definition of $\mathcal{L}_\Phi$ implies that for each $\sigma \in \Phi$ the (graded) $A$-module $\mathcal{L}_{\Phi,\sigma}$ has no negative or odd part (use induction on $d(\sigma)$). Thus the same is true for $H^0(\Phi; \mathcal{L}_\Phi)$ and 2) follows. Also by induction on $d(\sigma)$ (and using Lemma 3.11 above) one checks that the zero component of $\mathcal{L}_{\Phi,\sigma}$ has dimension 1. In other words the zero component of the sheaf $\mathcal{L}_\Phi$ is the constant sheaf $\mathbb{R}$. Thus by Lemma 3.11 $ih_0(\Phi) = 1$. □

We will be able to say much more about $IH(\Phi)$ in case $\Phi$ is a *complete* fan.

**Lemma 5.9.** *Let $\Psi$ be a subdivision of $\Phi$. Then $IH(\Phi)$ is a direct summand of $IH(\Psi)$. In particular, one has the inequality*

$$ih(\Psi) \geq ih(\Phi)$$

*(coefficient by coefficient) of polynomials with nonnegative coefficients.*

*Proof.* Let $\pi : (\Psi, \mathcal{A}_\Psi) \to (\Phi, \mathcal{A}_\Phi)$ denote the corresponding morphism of ringed spaces. Then $(\pi_* \mathcal{L}_\Psi)_{\underline{o}} = \mathbb{R}$. Hence the sheaf $(\pi_* \mathcal{L}_\Psi)$ contains $\mathcal{L}_\Phi$ as a direct summand (see Theorem 5.3). The lemma follows. □

## 6. Borel-Moore-Verdier duality

Let $\Phi$ be a fan in $V = \mathbb{R}^n$. Let $A_\Phi$ as usual be the constant sheaf on $\Phi$ with stalk $A$. Denote by $D^b_c(A_\Phi - mod)$ the bounded derived category of (locally finitely generated) $A_\Phi$-modules. In particular the additive category of sheaves $\mathfrak{M}(\mathcal{A}_\Phi)$ is a full subcategory of $D^b_c(A_\Phi - mod)$.

In this section we define the duality functor, i.e. a contravariant involution $\mathbb{D}$ on the category $D^b_c(A_\Phi - mod)$. The duality preserves the subcategory $\mathfrak{M}(\mathcal{A}_\Phi)$ and $\mathbb{D}(\mathcal{L}_\Phi) \simeq \mathcal{L}_\Phi$. Among other things, it gives rise to Poicaré duality in $IH(\Phi)$.

The contsruction of $\mathbb{D}$ follows the general pattern of Borel-Moore-Verdier duality. In particular, for any sheaf on $\Phi$ we define explicitly the co-sheaf of "compactly supported" sections.

1. **Co-sheaves and homology.** While a sheaf on a fan $\Phi$ (with values in an abelian category $\mathcal{C}$) is a functor $\Phi^0 \to \mathcal{C}$, a *co-sheaf* is a functor $\Phi \to \mathcal{C}$. A co-sheaf determines a functor $\Phi^0 \to \mathcal{C}^0$. Given a sheaf $F$ on $\Phi$ its space of global sections $\Gamma(\Phi; F)$ is by definition the inverse limit $\varprojlim_{\Phi} F$. This is a left exact functor. For a co-sheaf $\mathcal{V} : \Phi \to \mathcal{C}$ we define its space of global co-sections as the direct limit $\varinjlim_{\Phi} F$.

**Lemma 6.1.** *The functor of global co-sections is right exact.*

*Proof.* Let $A$ be a co-sheaf on $\Phi$ with coefficients in $\mathcal{C}$. The duality functor ${}^0 : \mathcal{C} \to \mathcal{C}^0$ makes it into a sheaf $A^0$ with coefficients if $\mathcal{C}^0$. We have $A^{00} = A$ and

$$(\varinjlim_{\Phi} A)^0 = \varprojlim_{\Phi} A^0.$$

Also the functor $A \mapsto A^0$ preserves exact sequences.

Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of co-sheaves. Then

$$0 \leftarrow A^0 \leftarrow B^0 \leftarrow C^0 \leftarrow 0$$

is also exact. Hence

$$\varprojlim_{\Phi} A^0 \leftarrow \varprojlim_{\Phi} B^0 \leftarrow \varprojlim_{\Phi} C^0 \leftarrow 0$$

is exact. Applying ${}^0$ to the last sequence we obtain the desired exact sequence

$$\varinjlim_{\Phi} A \to \varinjlim_{\Phi} B \to \varinjlim_{\Phi} C \to 0.$$

$\square$

*Remark* 6.2. Assume that the category $\mathcal{C}$ has enough projectives. Then so does the category of cosheaves on $\Phi$. Indeed, let $W$ be a projective object in $\mathcal{C}$. Choose a cone $\sigma \in \Phi$ and denote by $J$ the "constant co-sheaf" $W$ on $\mathrm{Star}(\sigma)$ "extended by zero" to $\Phi$. Then $J$ is a projective co-sheaf and every co-sheaf is a quotient of a direct sum of such.

Based on the above lemma and remark we may define the $i$-th homology of a co-sheaf $\mathcal{V}$ as the $i$-th left derived functor of global co-sections:

$$H_i(\Phi; \mathcal{V}) := H^{-i}\,\mathbf{L}\varinjlim_{\Phi} \mathcal{V}.$$

*Example* 6.3. Let $J$ be a co-sheaf as in the last remark. Then $H_0(\Phi; J) = W$ and $H_i(\Phi; J) = 0$, for $i > 0$.

2. **Co-sheaf of sections with compact support.** Fix an orientation of each cone $\sigma$ in $\Phi$. Let $F$ be a sheaf on $\Phi$. We want to define the complex $\Gamma_c(F)$ of co-sheaves of sections of $F$ with compact support. Let $\sigma \in \Phi$. Denote by $i_{[\sigma]} : [\sigma] \hookrightarrow \Phi$ the open embedding of the irreducible open set $[\sigma]$. Denote by $F_{[\sigma]}$ the extension by zero $i_{[\sigma]!} i^*_{[\sigma]} F$. Consider the cellular complex $C^\bullet(F_{[\sigma]})$ and put

$$\Gamma_c(F)_\sigma := C^\bullet(F_{[\sigma]})[n].$$

This is a complex which is concentrated in degrees $[-d(\sigma), 0]$. For $\tau \leq \sigma$ we have the inclusion of complexes $\Gamma_c(F)_\tau \hookrightarrow \Gamma_c(F)_\sigma$ which makes $\Gamma_c(F)$ a complex of co-sheaves.

The functor $\Gamma_c(\cdot)$ is exact, thus it extends trivially to the derived category of sheaves on $\Phi$. Let $\mathrm{Sh}(\Phi)$ as usual denote the category of sheaves of $\mathbb{R}$-vector spaces on $\Phi$.

**Proposition 6.4.** *Assume that the fan $\Phi$ is complete. The functors $\mathbf{L}\varinjlim_{\Phi}\Gamma_c(\cdot)$ and $\mathbf{R}\,\Gamma(\Phi;\cdot)[n]$ are naturally isomorphic (as functors from the bounded derived category $D^b(\mathrm{Sh}(\Phi))$ to $D^b(\mathbb{R}-vect)$).*

*Proof.* Fix a sheaf $F$ on $\Phi$. Then $\Gamma_c(F)$ is a complex which consists of direct sums of projective co-sheaves $J$ as in Example 6.3 above. Thus the natural morphism

$$\mathbf{L}\varinjlim \Gamma_c(F) \longrightarrow \varinjlim \Gamma_c(F)$$

is an isomorphism. On the other hand $\varinjlim \Gamma_c(F)$ is just the shifted cellular complex $C^\bullet(F)[n]$. Since the fan $\Phi$ is complete $C^\bullet(F) = \mathbf{R}\,\Gamma(\Phi;F)$. This proves the proposition. □

3. **Duality in** $D^b_c(A_\Phi - mod)$**.** We are going to define a version of Borel-Moore-Verdier duality for sheaves of $A_\Phi$-modules. Let us first fix some notation.

Define the dualizing $A$-module as follows

$$\omega = \omega_{A/\mathbb{R}} := A \otimes \det V^*.$$

Thus $\omega$ is a free $A$-module of rank one generated in degree $2\dim V$, i.e. $\omega \simeq A(-2n)$. For a complex $M$ of $A$-modules denote by $M^*$ the complex $\mathrm{Hom}^\bullet_A(M,\omega)$. In what follows we will omit the subscript $A$ (resp. $A_\Phi$) when talking about morphisms of $A$-modules (resp. $A_\Phi$-modules).

**Definition 6.5.** Fix $F \in D^b_c(A_\Phi - mod)$. The category of $A_\Phi$-modules has enough projective objects. Let $P(F) \to F$ be a projective resolution of $F$. (We may assume that $P(F)$ is a finite complex if projectives $A_\Phi$-modules). Then in particular for each cone $\sigma \in \Phi$ the stalk $P(F)_\sigma$ is a complex of free $A$-modules. Consider the complex of co-sheaves of $A$-modules $\Gamma_c(P(F))$. Finally put

$$\mathbb{D}(F) := \Gamma_c(P(F))^*[n].$$

This is a complex of sheaves of $A$-modules, whose stalk at $\sigma$ is the complex

$$\mathbb{D}(F)_\sigma = \Gamma_c(P(F))^*_\sigma[n] = C^\bullet(F_{[\sigma]})^*.$$

Thus if $P(F)$ is a single sheaf in degree zero, then $\mathbb{D}(F)_\sigma$ is a complex concentrated in degrees $[-n, -n + d(\sigma)]$.

*Remark* 6.6. Note that the duality $\mathbb{D}$ is a local functor, i.e. it commutes with restriction of sheaves to subfans.

**Proposition 6.7.** *Let $\Phi$ be a fan in $V$. There is a natural isomorphism of functors from $D^b_c(A_\Phi - mod)$ to $D^b(A - mod)$:*

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(\cdot)) \simeq \mathbf{R}\,\mathrm{Hom}(C^\bullet(\cdot),\omega).$$

*Proof.* Let $P$ be a projective $A_\Phi$-module. Then $\mathbb{D}(P)$ is a complex of sheaves with the following graded components

$$\mathbb{D}(P)^{-n+i} = \bigoplus_{d(\sigma)=i} (P^*_\sigma)_{\mathrm{Star}(\sigma)},$$

where $(P^*_\sigma)_{\mathrm{Star}(\sigma)}$ is the extension by zero to $\Phi$ of the constant sheaf on $\mathrm{Star}(\sigma)$ with stalk $\mathrm{Hom}(P_\sigma,\omega)$. In particular, such sheaves are injective (when considered as sheaves of vector spaces) and hence

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(P)^{-n+i}) = \Gamma(\Phi;\mathbb{D}(P)^{-n+i}) = \bigoplus_{d(\sigma)=i} P^*_\sigma.$$

Thus the complex of global sections $\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(P))$ is just the dual of the cellular complex $C^\bullet(P)$:

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(P)) = C^\bullet(P)^*.$$

The last formula remains true if $P$ is a complex of projective $A_\Phi$-modules.

Now fix $F \in D^b_c(A_\Phi - mod)$ and let $P(F) \to F$ be its projective resolution. Since the cellular complex is an exact functor we have the quasi-isomorphism

$$C^\bullet(P(F)) \simeq C^\bullet(F).$$

Note that the complex $C^\bullet(P(F))$ consists of free $A$-modules, so it is a projective resolution of $C^\bullet(F)$. Thus

$$\mathbf{R}\,\mathrm{Hom}(C^\bullet(F),\omega) = \mathrm{Hom}(C^\bullet(P(F)),\omega) = C^\bullet(P(F))^*.$$

Combining this with the equality

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(F)) = \mathbf{R}\,\Gamma(\Phi;\mathbb{D}(P(F))) = C^\bullet(P(F))^*$$

we obtain the desired isomorphism. □

**Corollary 6.8.** *Let $\Phi$ be a fan in $V$. Assume that $\Phi$ is complete. Then there is a natural isomorphism of functors from $D^b_c(A_\Phi - mod)$ to $D^b(A - mod)$:*

$$\mathbf{R}\,\Gamma(\Phi;\mathbb{D}(\cdot)) \simeq \mathbf{R}\,\mathrm{Hom}(\mathbf{R}\,\Gamma(\Phi;\cdot),\omega).$$

*Proof.* This follows from the last proposition and Proposition 3.3 □

Later on we will need the following version of the above corollary.

**Corollary 6.9.** *Let $\Phi$ be a fan in $V$ and $\sigma \in \Phi$ be a cone. Consider the subfan $\Psi := [\partial\sigma] \subset \Phi$. Then there is a natural isomorphism of functors from $D^b_c(A_\Phi - mod)$ to $D^b(A - mod)$:*

$$\mathbf{R}\,\Gamma(\Psi; \mathbb{D}(\cdot)) \simeq \mathbf{R}\,\mathrm{Hom}(\mathbf{R}\,\Gamma(\Psi; \cdot), \omega)[n - d(\sigma) + 1].$$

*Proof.* This follows from Proposition 6.7, Remark 6.6 and Corollary 3.5. ☐

We are going to give an alternative description of the duality functor (as in the usual case of sheaves on locally compact spaces).

**Definition 6.10.** Define the *dualizing complex* $D_\Phi$ on the fan $\Phi$ as

$$D_\Phi := \mathbb{D}(A_\Phi).$$

The complex $D_\Phi$ is concentrated in degrees $[-n, o]$ and $D_\Phi^{-n+i}$ is the direct sum over all cones $\tau$ of dimension $i$ of constant sheaves with stalk $A^*_\tau = \omega$ on the closed set $\mathrm{Star}(\tau)$ extended by zero to $\Phi$.

**Proposition 6.11.** *There is a canonical isomorphism of contravariant endofunctors of $D^b_c(A_\Phi - mod)$:*

$$\mathbb{D}(\cdot) = \mathbf{R}\,\mathcal{H}om(\cdot, D_\Phi).$$

*Proof.* Suppose that $F \in D^b_c(A_\Phi - mod)$ is a complex of projective $A_\Phi$-modules. We are going to show that complexes of $A_\Phi$-modules $\mathbb{D}(F)$ and $\mathcal{H}om(F, D_\Phi)$ are naturally isomorphic. To simplify the notation assume that $F$ is a single sheaf in degree 0. Fix $\sigma \in \Phi$. Then

$$\begin{aligned}
\mathcal{H}om^{i-n}(F, D_\Phi)_\sigma &= \mathrm{Hom}^{i-n}(F_{[\sigma]}, D_\Phi) \\
&= \mathrm{Hom}(F_{[\sigma]}, D_\Phi^{i-n}) \\
&= \bigoplus_{\substack{d(\tau)=i \\ \tau \le \sigma}} F^*_\tau \\
&= \mathbb{D}(F)^{i-n}_\sigma.
\end{aligned}$$

This identification is compatible with the differential in the two complexes. This proves the proposition. ☐

In fact the dualizing complex $D_\Phi$ is very simple. Namely, consider $\omega$ as a constant sheaf on the open point $\underline{o} \in \Phi$. Let $\omega_{\underline{o}}$ denote its extension by zero to $\Phi$. Note that the stalk $D_{\Phi,\underline{o}}$ is isomorphic to $\omega_{\underline{o}}[n]$.

**Lemma 6.12.** *The natural map*

$$\omega_{\underline{o}}[n] \to D_\Phi$$

*is a quasi-isomorphism.*

*Proof.* The lemma claims that for each $\sigma \neq \underline{o}$ the complex $D_{\Phi,\sigma}$ is acyclic. This complex is $C^\bullet(A_{[\sigma]})^*$ and $C^\bullet(A_{[\sigma]})$ is acyclic by Lemma 3.7. □

4. **Biduality.** Our next goal is to prove Theorem 6.23 below. Let us begin with some preparations.

**Definition 6.13.** A nonempty open subset $U \subset \Phi$ is *saturated* if whenever the boundary of a cone is in $U$, then the cone itself is in $U$.

*Example* 6.14. $\Phi$ and $\{\underline{o}\}$ are saturated.

**Definition 6.15.** For a nonempty open subset $U \subset \Phi$ define its *opposite* $U'$ as follows:

$$U' := \{\sigma \in \Phi | \forall \tau \in U, \quad \tau \cap \sigma = \underline{o}\}.$$

*Remark* 6.16. 1. $\Phi' = \{\underline{o}\}$, $\{\underline{o}\}' = \Phi$.
2. For any $U$ its opposite $U'$ is saturated.
3. We have $U \subset U''$.

**Lemma 6.17.** *If $U$ is saturated then $U = U''$.*

*Proof.* Let $\underline{o} \neq \sigma \in U''$. Let $\tau \leq \sigma$ be a face of dimension 1. Then $\tau \notin U'$. Hence $\tau \in U$. Thus by induction on dimension all faces of $\sigma$ are in $U$ and so $\sigma$ is in $U$. □

**Corollary 6.18.** *The map $U \mapsto U'$ is an involution of the collection of saturated open subsets of $\Phi$.*

For an open subset $U \subset \Phi$ and an $A$-module $M$ denote as usual by $M_U$ the extension by zero to $\Phi$ of the constant sheaf $M$ on $U$. In case $U = \{\underline{o}\}$ we will also denote this sheaf by $M_{\underline{o}}$.

*Remark* 6.19. Let $U \subset \Phi$ be open. Note the equality of sheaves

$$\mathcal{H}om(A_U, \omega_{\underline{o}}) = \omega_{U'}.$$

Hence if $U$ is saturated, then by Lemma 6.17 the obvious map

$$A_U \to \mathcal{H}om(\mathcal{H}om(A_U, \omega_{\underline{o}}), \omega_{\underline{o}}),$$

$$a \mapsto (f \mapsto f(a))$$

is an isomorphism.

**Definition 6.20.** Let us introduce the apropriate version of the cocellular complex (see 3.14) for $A_\Phi$-modules. For a cone $\sigma \in \Phi$ let $i_\sigma : \{\sigma\} \hookrightarrow \Phi$ denote the inclusion. Consider the dualizing module $A$-module $\omega$ as a sheaf on the point $\sigma$. Then the $A_\Phi$-module $i_{\sigma *}\omega$ is a constant sheaf on $\mathrm{Star}(\sigma)$ with stalk $\omega$. If $\tau \leq \sigma$ then there is a natural surjection of sheaves $r_{\tau\sigma} : i_{\tau *}\omega \to i_{\sigma *}\omega$. Put

$$K_A^{-n+j} := \bigoplus_{d(\sigma)=j} i_{\sigma *}\omega.$$

As usual, the maps $r_{\tau\sigma}$ with the sign $\pm 1$ define the differential in the complex

$$K_A^\bullet := K^{-n} \to K^{-n+1} \to \ldots \to K^0.$$

This complex is a resolution of the dualizing sheaf $\omega_{\underline{o}}[n]$ (the proof is the same as that of Lemma 3.13).

Given an $A_\Phi$-module $F$ define its *cocellular complex* $K_A^\bullet(F)$ as

$$K_A^\bullet(F) := \mathrm{Hom}^\bullet(F, K_A^\bullet).$$

*Remark* 6.21. 1. Note that $\mathrm{Hom}(F, i_{\sigma *}\omega) = \mathrm{Hom}(F_\sigma, \omega)$. Hence if stalks $F_\sigma$ are free $A$-modules, then

$$K_A^\bullet(F) = \mathbf{R}\,\mathrm{Hom}(F, \omega_{\underline{o}}[n]).$$

2. The cocellular complex $K_A^\bullet(F)$ is just the dual of the cellular complex $C^\bullet(F)$:

$$K_A^\bullet(F) = \mathrm{Hom}^\bullet(C^\bullet(F), \omega) = C^\bullet(F)^*.$$

**Lemma 6.22.** *Let $\sigma \in \Phi$ and put $U := [\sigma]'$. Then*

$$\mathcal{H}om(A_U, \omega_{\underline{o}}) = \mathbf{R}\,\mathcal{H}om(A_U, \omega_{\underline{o}}).$$

*Proof.* Fix $\tau \in \Phi$. We need to show that the stalk $\mathbf{R}^i\,\mathcal{H}om(A_U, \omega_{\underline{o}})_\tau = 0$ for $i > 0$. This is the same as showing that $\mathbf{R}^i\,\mathrm{Hom}(A_W, \omega_{\underline{o}}) = 0$, for $i > 0$, where $W = U \cap [\tau]$. This is trivial if $\tau = \underline{o}$, so assume that $\tau \neq \underline{o}$. By Remark 6.21(1) above it suffices to show that the cocellular complex $K_A^\bullet(A_W)$ is acyclic in degrees $> -n$. If $W = \underline{o}$, then $K_A^i(A_W) = 0$ if $i > -n$. So assume that $W \neq \underline{o}$. Then the space $\tau - W$ is contractible. Therefore the cocellular complex $K_A^\bullet(A_W)$ is isomorphic to the augmented (shifted) cochain complex of a contractible space; hence it is acyclic. ∎

We are ready to prove the main result of this section.

**Theorem 6.23.** *Let $\Phi$ be a fan in $V$. The duality $\mathbb{D}$ is an anti-involution of the category $D^b_c(A_\Phi - mod)$. More precisely, there exists an isomorphism of functors*

$$Id \xrightarrow{\sim} \mathbb{D} \circ \mathbb{D}.$$

*Proof.* For $F \in D^b_c(A_\Phi - mod)$ we have a functorial morphism

$$\alpha : F \to \mathbb{D} \circ \mathbb{D}(F) = \mathbf{R}\,\mathcal{H}om(\mathbf{R}\,\mathcal{H}om(F, D_\Phi), D_\Phi),$$

$$a \mapsto (f \mapsto f(a)).$$

Let $\sigma \in \Phi$. Then $A_{[\sigma]}$ is a projective $A_\Phi$ -module. It suffices to prove that $\alpha(A_{[\sigma]})$ is a quasi-isomorphism for all $\sigma$.

By Remark 6.19 and Lemma 6.22 we have

$$\begin{aligned} \mathbb{D} \circ \mathbb{D}(A_{[\sigma]}) &= \mathbf{R}\,\mathcal{H}om(\mathcal{H}om(A_{[\sigma]}, \omega_{\underline{o}}[n]), \omega_{\underline{o}}[n]) \\ &= \mathbf{R}\,\mathcal{H}om(\omega_{[\sigma]'}, \omega_{\underline{o}}) \\ &= \mathcal{H}om(\omega_{[\sigma]'}, \omega_{\underline{o}}) \\ &= A_{[\sigma]}. \end{aligned}$$

Hence $\alpha(A_{[\sigma]})$ is an isomorphism by Remark 6.19. This proves the theorem. □

5. **Poincare duality for $IH(\Phi)$.**

**Proposition 6.24.** *Let $\Phi$ be a fan in $V$. The duality $\mathbb{D}$ preserves the subcategory $\mathfrak{M}(\mathcal{A}_\Phi)$.*

*Proof.* Fix $\mathcal{M} \in \mathcal{M}(\mathcal{A}_\Phi)$. We need to show

a) $\mathbb{D}(\mathcal{M})$ is quasi-isomorphic to a single sheaf (sitting in degree 0).

b) The $A_\Phi$-module $\mathbb{D}(\mathcal{M})$ is actually an $\mathcal{A}_\Phi$-module and as such is locally free.

c) The sheaf $\mathbb{D}(\mathcal{M})$ is flabby.

a), b): Since the sheaf $\mathcal{M}$ is flabby the cohomology of the complex $\Gamma_c(\mathcal{M}_{[\sigma]})$ vanishes except in degree $-d(\sigma)$ (see Lemma 3.9). Put $K := H^{-d(\sigma)}\Gamma_c(\mathcal{M}_{[\sigma]})$; it is a free $\mathcal{A}_{\Phi,\sigma}$-module. This means that $\mathbf{R}^i \operatorname{Hom}_A(\Gamma_c(\mathcal{M}_{[\sigma]}), \omega[n]) = 0$ for $i \neq 0$ and $\mathbf{R}^0 \operatorname{Hom}_A(\Gamma_c(\mathcal{M}_{[\sigma]}), \omega[n]) = \operatorname{Ext}^i_A(K, \omega)$ is a free $\mathcal{A}_{\Phi,\sigma}$-module. This proves a) and b).

It remains to prove that the restriction map $\mathbb{D}(\mathcal{M})_\sigma \to \Gamma(\partial\sigma, \mathbb{D}(\mathcal{M}))$ is surjective (Lemma 3.1). Consider the open set $[\partial\sigma]$ and the cellular complex $C^\bullet(\mathcal{M}_{[\partial\sigma]})$. By Corollary 3.5 it is quasi-isomorphic to $\mathbf{R}\,\Gamma(\partial\sigma; \mathcal{M})[-n+d(\sigma)-1]$. Since $\mathcal{M}$ is flabby we have $H^j(\partial\sigma; \mathcal{M}) = 0$ unless $j = 0$. Put $N := H^0(\partial\sigma; \mathcal{M})$. Since $\mathcal{M}$ is locally free we have $\operatorname{Ext}^j_A(N, A) = 0$ unless $j = n - d(\sigma) + 1$. Moreover by Corollary 6.9

$$\Gamma(\partial\sigma; \mathbb{D}(\mathcal{M})) = \mathbf{R}\operatorname{Hom}(\mathbf{R}\,\Gamma(\partial\sigma; \mathcal{M}), \omega)[n - d(\sigma) + 1].$$

Thus $\Gamma(\partial\sigma; \mathbb{D}(\mathcal{M})) = \operatorname{Ext}^{n-d(\sigma)+1}(N, \omega)$.

We have the exact sequence of $A$-modules

$$0 \to K \to \mathcal{M}_\sigma \to N \to 0,$$

which gives rise to the exact sequence

$$0 \to \mathrm{Ext}_A^{n-d(\sigma)}(\mathcal{M}_\sigma, \omega) \to \mathrm{Ext}_A^{n-d(\sigma)}(K, \omega) \xrightarrow{f} \mathrm{Ext}_A^{n-d(\sigma)+1}(N, \omega) \to 0.$$

The map $f$ coincides with the restriction map

$$\mathbb{D}(\mathcal{M})_\sigma \to \Gamma(\partial\sigma; \mathbb{D}(\mathcal{M})),$$

hence the latter is surjective. This proves c).

□

**Corollary 6.25.** *Let $\Phi$ be a fan in $\mathbb{R}^n$. The duality functor induces an anti-involution of the category $\mathfrak{M}(\mathcal{A}_\Phi)$.*

*Proof.* Since $\mathbb{D}$ is an anti-involution of $D^b_c(A_\Phi - mod)$ and it preserves $\mathfrak{M}(\mathcal{A}_\Phi)$ the corollary follows. □

**Corollary 6.26.** *Let $\Phi$ be a fan in $\mathbb{R}^n$ and $\sigma \in \Phi$. Let $\mathcal{L}^\sigma_\Phi(k)$ be the minimal sheaf on $\Phi$ based at $\sigma$. Then $\mathbb{D}(\mathcal{L}^\sigma_\Phi(k)) \simeq \mathcal{L}^\sigma_\Phi(-k-2d(\sigma))$. In particular, $\mathbb{D}(\mathcal{L}_\Phi) \simeq \mathcal{L}_\Phi$.*

*Proof.* By the above corollary the duality sends indecomposable objects of $\mathfrak{M}(\mathcal{A}_\Phi)$ to indecomposable ones. We have $\mathrm{Supp}(\mathbb{D}(\mathcal{L}^\sigma_\Phi(k))) \subset \mathrm{Star}(\sigma)$. Also $\mathbb{D}(\mathcal{L}^\sigma_\Phi(k))_\sigma = \mathrm{Ext}_A^{n-d(\sigma)}(\mathcal{A}_{\Phi,\sigma}(k), \omega) \simeq \mathcal{A}_{\Phi,\sigma}(-k-2d(\sigma))$ (see the proof of Proposition 6.24 above). Thus the corollary follows from Proposition 5.2(3) and Theorem 5.3. □

**Corollary 6.27.** *Let $\Phi$ be a complete fan in $\mathbb{R}^n$. Then there exists an isomorphism of $A$-modules:*

$$\Gamma(\Phi; \mathcal{L}_\Phi) \simeq \mathrm{Hom}(\Gamma(\Phi; \mathcal{L}_\Phi), \omega),$$

*i.e. the free $A$-module $\Gamma(\Phi; \mathcal{L}_\Phi)$ is self-dual.*

*Proof.* The minimal sheaf $\mathcal{L}_\Phi$ is flabby, so $H^i(\Phi; \mathcal{L}_\Phi) = 0$ for $i > 0$. By Theorem 4.7 the $A$-module $\Gamma(\Phi; \mathcal{L}_\Phi)$ is free. The above corollary implies that the same is true for $\mathbb{D}(\mathcal{L}_\Phi)$. Hence the natural isomorphism of Proposition 6.8

$$\mathbf{R}\,\Gamma(\Phi; \mathbb{D}(\mathcal{L}_\Phi)) = \mathbf{R}\,\mathrm{Hom}(\mathbf{R}\,\Gamma(\Phi; \mathcal{L}_\Phi), \omega)$$

reduces to

$$\Gamma(\Phi; \mathbb{D}(\mathcal{L}_\Phi)) = \mathrm{Hom}(\Gamma(\Phi; \mathcal{L}_\Phi), \omega).$$

A choice of an isomorphism of $\mathbb{D}(\mathcal{L}_\Phi)$ and $\mathcal{L}_\Phi$ provides an isomorphism of $A$-modules

$$\Gamma(\Phi; \mathcal{L}_\Phi) \simeq \mathrm{Hom}(\Gamma(\Phi; \mathcal{L}_\Phi), \omega).$$

□

Consider the 1-dimensional (graded) vector space $\overline{\omega}$. It has degree $2n$.

**Corollary 6.28.** *Let $\Phi$ be a complete fan in $\mathbb{R}^n$. There exists an isomorphism of graded vector spaces*

$$IH(\Phi) \simeq \mathrm{Hom}_{\mathbb{R}}(IH(\Phi), \overline{\omega}).$$

*Proof.* Immediate from the previous corollary. □

**Corollary 6.29.** *Let $\Phi$ be a complete fan in $\mathbb{R}^n$. Then*

1. *$ih_{n-j}(\Phi) = ih_{n+j}(\Phi)$ for all $j$.*
2. *$ih_j(\Phi) = 0$ unless $j$ is even and $j \in [0, 2n]$.*
3. *$ih_0(\Phi) = 1 = ih_{2n}(\Phi)$.*

*Proof.* The first assertion is immediate from the last corollary. Recall that $ih_0(\Phi) = 1$ and $ih_j(\Phi) = 0$ if $j < 0$ or $j$ odd (Lemma 5.7). This implies the last two assertions. □

## 7. Toward Hard Lefschetz and the combinatorial invariance

Throughout this section $\Phi$ will denote a complete fan in a vector space $V$ of dimension $n$.

1. **Ampleness in the context of fans.** Consider the short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_\Phi \longrightarrow V^* \longrightarrow \mathcal{G} \longrightarrow 0\ ,$$

where $V^*$ denotes the constant sheaf and $\Omega^1_{\Phi,\sigma} = \mathrm{Span}(\sigma)^\perp$. Since constant sheaves have trivial higher cohomology and $\Omega^1_\Phi$ is supported on $\Phi_{\leq n-1}$, the long exact sequence in cohomology reduces, in low degrees, to the short exact sequence of vector spaces

$$0 \longrightarrow V^* \longrightarrow \Gamma(\Phi, \mathcal{G}) \longrightarrow H^1(\Phi; \Omega^1_\Phi) \longrightarrow 0\ .$$

The space $\Gamma(\Phi; \mathcal{G})$ consists of continuous, cone-wise linear functions on $\Phi$.

For any object $\mathcal{M}$ of $\mathfrak{M}(\mathcal{A}_\Phi)$, the elements of $\Gamma(\Phi; \mathcal{G})$ act naturally on the free graded $A$-module $\Gamma(\Phi; \mathcal{M})$ by endomorphisms of degree two. Clearly, the induced action on the graded vector space $\overline{\Gamma(\Phi; \mathcal{M})}$ factors through $H^1(\Phi; \Omega^1_\Phi)$.

**Definition 7.1.** An element $\bar{l}$ of $H^1(\Phi;\Omega^1_\Phi)$ is called *ample* iff it admits a lifting $l \in \Gamma(\Phi;\mathcal{M})$ which is strictly convex.

2. **Hard Lefschetz for complete fans.** The statement of Conjecture 7.2 (below) is the analog of the Hard Lefschetz Theorem in the present context. Recall that $\mathcal{L}_\Phi$ denotes the indecomposable object of $\mathfrak{M}(\mathcal{A}_\Phi)$ based at the origin, with $\mathcal{L}_{\Phi,\underline{o}} = \mathbb{R}$. For a graded vector space $W$ we will denote by $W^{(i)}$ the subspace of homogeneous elements of degree $i$.

**Conjecture 7.2** (Hard Lefschetz Conjecture). *An ample $\bar{l} \in H^1(\Phi;\Omega^1_\Phi)$ induces a Lefschetz operator on the graded vector space $IH(\Phi)$, i.e. for every $i$ the map $\bar{l}^i : IH(\Phi)^{(n-i)} \longrightarrow IH(\Phi)^{(n+i)}$ is an isomorphism.*

For a rational fan $\Phi$ this conjecture follows immediately from results in [BL], ch.15.

The above conjecture has the following standard corollary.

**Corollary 7.3.** *Assume the Hard Lefschetz Conjecture. Then for an ample $\bar{l}$ the map $\bar{l} : IH(\Phi)^{(i)} \longrightarrow IH(\Phi)^{(i+2)}$ is injective for $i \leq n-1$ and surjective for $i \geq n-1$. In particular*

$$ih_0(\Phi) \leq ih_2(\Phi) \leq ... \leq ih_{2[n/2]}(\Phi).$$

3. **Local intersection cohomology.** Let $\Psi$ be a fan in $V$ and $\sigma \in \Psi$.

**Definition 7.4.** Define the *local intersection cohomology* space

$$IP(\sigma) := \overline{\mathcal{L}_{\Psi,\sigma}}$$

and denote by $ip(\sigma)$ the corresponding Poincare polynomial.

*Remark* 7.5. 1. Consider the subfan $[\sigma] \subset \Psi$. Then by Proposition 5.2(4) $\mathcal{L}_\Psi|_{[\sigma]} = \mathcal{L}_{[\sigma]}$. Thus in particular the local intersection cohomology $IP(\sigma)$ depends only on the cone $\sigma$ and not on the fan $\Psi$.

2. Note that $ip_j(\sigma) = 0$ if $j$ is odd or negative.

4. **The global-local formula.** For a cone $\sigma \subset V$ of dimension $d(\sigma) = d+1 \geq 2$ consider the subspace $W := \mathrm{Span}(\sigma) \subset V$. Choose a linear isomorphism

$$W \simeq \mathbb{R}^d \times \mathbb{R}$$

so that the ray $(0,\mathbb{R}^+)$ lies in the interior of $\sigma$. This defines the projection $p$ of $\sigma$ to $\mathbb{R}^d$ so that the image $\overline{\partial\sigma} := p(\partial\sigma)$ is a complete fan in $\mathbb{R}^d$. Also $\partial\sigma$ becomes the graph of a function $l : \mathbb{R}^d \to \mathbb{R}$ which is piecewise linear and strictly convex with respect to the fan $\overline{\partial\sigma}$. In particular $\bar{l} \in H^1(\overline{\partial\sigma};\Omega^1_{\overline{\partial\sigma}})$ is an ample class.

Note that $A_W = A_{\mathbb{R}^d}[l]$. Let $m_1 \subset A_W$ and $m_2 \subset A_{\mathbb{R}^d}$ denote the maximal ideals, so that $m_1 = m_2[l]$.

The minimal sheaf $\mathcal{L}_{[\partial\sigma]}$ as the $\mathcal{A}_{[\partial\sigma]}$-module is obtained by extension of scalars

$$\mathcal{L}_{[\partial\sigma]} = \mathcal{A}_{[\partial\sigma]} \otimes_{\mathcal{A}_{\overline{\partial\sigma}}} \mathcal{L}_{\overline{\partial\sigma}}.$$

Thus the intersection cohomology $IH(\overline{\partial\sigma})$ is an $\mathbb{R}[l]$-module. We have

$$IH(\overline{\partial\sigma})/l \cdot IH(\overline{\partial\sigma}) \simeq \Gamma(\partial\sigma; \mathcal{L}_{[\partial\sigma]})/m_1\Gamma(\partial\sigma; \mathcal{L}_{[\partial\sigma]}) \simeq \overline{\mathcal{L}_{[\sigma],\sigma}} = IP(\sigma).$$

The Hard Lefschetz Conjecture (for $\overline{\partial\sigma}$) implies that $\bar{l}$ is a Lefschetz operator on $IH(\overline{\partial\sigma})$. Thus $IP(\sigma)$ is isomorphic to the $l$-primitive part of $IH(\overline{\partial\sigma})$. In particular the Poincare polynomial $ih(\overline{\partial\sigma})$ depends only on $\sigma$ and not on a particular choice of the isomorphism

$$\operatorname{Span}(\sigma) \simeq \mathbb{R}^d \times \mathbb{R}.$$

Let us summarize our discussion in the following corollary.

**Corollary 7.6.** *Let $\sigma \subset V$ be a cone of dimension $d+1 \geq 2$. Choose an isomorphism*

$$\operatorname{Span}(\sigma) \simeq \mathbb{R}^d \times \mathbb{R}$$

*as above, so that the projection $\overline{\partial\sigma}$ of the fan $[\partial\sigma]$ is a complete fan in $\mathbb{R}^d$. Then the Hard Lefschetz Conjecture implies that*

*1. The Poincare polymonial $ih(\overline{\partial\sigma})$ is independent of the choices made.*

*2.*

$$ip_j(\sigma) = \begin{cases} ih_j(\overline{\partial\sigma}) - ih_{j-2}(\overline{\partial\sigma}) & \text{for } 0 \leq j \leq d \\ 0 & \text{otherwise} \end{cases}$$

**Definition 7.7.** In the above corollary denote the Poincare polynomial $ih(\overline{\partial\sigma})$ by $ih(\sigma)$. We have

$$ip_j(\sigma) = \begin{cases} ih_j(\sigma) - ih_{j-2}(\sigma) & \text{for } 0 \leq j \leq d \\ 0 & \text{otherwise} \end{cases}$$

We call the last equation the *global-local formula.* In case $\sigma$ has dimension 0 or 1 put $ih(\sigma) = 1$. (Note that in this case $ip(\sigma) = 1$.) Note also that if cones $\sigma$ and $\sigma'$ are linearly isomorphic, then $ip(\sigma) = ip(\sigma')$ and $ih(\sigma) = ih(\sigma')$.

5. **The local-global formula.** In this section we express the Poincaré polynomial $ih(\Phi)$ of a complete fan $\Phi$ in terms of the local Poincaré polynomials $ip(\sigma)$ for $\sigma \in \Phi$. The argument is standard and is independent of any conjectures.

**Proposition 7.8.** *For a complete fan $\Phi$ in $\mathbb{R}^n$ we have the following relation between Poincaré polynomials in the variable $q$:*

$$ih(\Phi)(q) = \sum_{\sigma\in\Phi}(q^2-1)^{n-\dim\sigma} ip(\sigma)(q).$$

*Proof.* The quasi-isomorphism (Proposition 3.3)

$$\Gamma(\Phi;\mathcal{L}_\Phi) \to C^\bullet(\mathcal{L}_\Phi)$$

induces the quasi-isomorphism

$$\Gamma(\Phi;\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R} \to C^\bullet(\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R}$$

and the equality of the graded Euler characteristics

$$\chi(\Gamma(\Phi;\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R}) = \chi(C^\bullet(\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R})\ .$$

Since $\Gamma(\Phi;\mathcal{L}_\Phi)$ is free over $A$, the canonical map

$$\Gamma(\Phi;\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R} \to \Gamma(\Phi;\mathcal{L}_\Phi)\otimes_A \mathbb{R} = IH(\Phi)$$

is a quasi-isomorphism and

$$\chi(\Gamma(\Phi;\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R}) = ih(\Phi)\ .$$

Since $C^\bullet(\mathcal{L}_\Phi))$ is a complex of finitely generated $A$-modules and $A$ has finite Tor-dimension it follows that

$$\chi(C^\bullet(\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R}) = \sum_i (-1)^i \chi(C^i(\mathcal{L}_\Phi)\otimes^{\mathbb{L}}_A \mathbb{R})$$

Since $C^i(\mathcal{L}_\Phi)$ is isomorphic to $\bigoplus_{\dim\sigma=n-i}\mathcal{L}_{\Phi,\sigma}$ the above formulas imply the equality

$$ih(\Phi) = \sum_{\sigma\in\Phi}(-1)^{n-\dim\sigma}\chi(\mathcal{L}_{\Phi,\sigma}\otimes^{\mathbb{L}}_A \mathbb{R})\ .$$

By definition of $\mathcal{L}_\Phi$, the stalk $\mathcal{L}_{\Phi,\sigma}$ is a free module over $\mathcal{A}_{\Phi,\sigma}$ of graded rank $ip(\sigma)$. The standard caclulation with the Koszul complex shows that $\mathcal{A}_{\Phi,\sigma}\otimes^{\mathbb{L}}_A \mathbb{R}$ is represented by the complex (with trivial differential) $\bigoplus_i \bigwedge^i \sigma^\perp$. It follows that

$$\chi(\mathcal{L}_{\Phi,\sigma}\otimes^{\mathbb{L}}_A \mathbb{R}) = (1-q^2)^{n-\dim\sigma} ip(\sigma)$$

and

$$ih(\Phi) = \sum_{\sigma\in\Phi}(-1)^{n-\dim\sigma}(1-q^2)^{n-\dim\sigma} ip(\sigma) = \sum_{\sigma\in\Phi}(q^2-1)^{n-\dim\sigma} ip(\sigma)$$

□

6. **Summary.** Assuming the Hard Lefschetz Conjecture for complete fans we have associated two polynomials $ip(\sigma)$ and $ih(\sigma)$ to any cone $\sigma \subset V$. The odd coefficients of these polynomials vanish and the following relations hold (here $d+1 := d(\sigma)$):

1. $ip(\underline{o}) = ih(\underline{o}) = 1$.
2. $ip_j(\sigma) = \begin{cases} ih_j(\sigma) - ih_{j-2}(\sigma) & \text{for } 0 \leq j \leq d \\ 0 & \text{otherwise} \end{cases}$
3. $ih(\sigma)(q) = \sum\limits_{\tau < \sigma} (q^2-1)^{d-d(\tau)} ip(\tau)(q)$.

Indeed, the first two relations are contained in Definition 7.7 and the third one follows from Proposition 7.8 applied to the complete fan $\overline{\partial\sigma}$ as in Corollary 7.6 above.

As an immediate consequence of the above relations we obtain (by induction on the dimension $d(\sigma)$) that the polynomials $ip(\sigma)$ and $ih(\sigma)$ are combinatorial invariants of $\sigma$, i.e. they depend only on the face lattice of $\sigma$.

Recall that in case $d > 0$ the polynomial $ih(\sigma)$ is defined as $ih(\overline{\partial\sigma})$ for a complete fan $\overline{\partial\sigma}$ of dimension $d$. Hence it follows from Corollary 6.29 and Corollary 7.3 that

1. $ih_0(\sigma) = 1 = ih_{2d}(\sigma)$.
2. $ih_j(\sigma) = 0$, unless $j$ is even and $j \in [0, 2d]$.
3. For all $j$ $ih_{d-j}(\sigma) = ih_{d+j}(\sigma)$.
4. $ih_0(\sigma) \leq ih_2(\sigma) \leq ... \leq ih_{2[d/2]}(\sigma)$.

7. **The $h$-vector and Stanley's conjectures.** Let $Q \subset \mathbb{R}^n$ be a convex polytope of dimension $d$. In [S] Stanley defined two polynomials $g(Q)$ and $h(Q)$. These polynomials are defined simultaneously and recursively for faces of $Q$, including the empty face $\emptyset$, as follows:

1. $g(\emptyset) = h(\emptyset) = 1$.
2. $g_j(Q) = \begin{cases} h_j(Q) - h_{j-1}(Q) & \text{for } 0 \leq j \leq [d/2] \\ 0 & \text{otherwise} \end{cases}$
3. $h(Q)(t) = \sum_{P \subset Q} (t-1)^{d-d(P)-1} g(P)(t)$, where the last summation is over all proper faces $P$ of $Q$ including the empty face $\emptyset$. Here $d(P)$ is the dimension of $P$ and $d(\emptyset) = -1$.

Stanley proved (in a more general context of Eulerian posets) the "Poincaré duality" for $h(Q)$:

$$h_j = h_{d-j},$$

and conjectured that

$$0 \leq h_0 \leq h_1 \leq ... \leq h_{[d/2]}.$$

Let us show how this conjecture follows from the Hard Lefschetz Conjecture. Namely, consider the space $\mathbb{R}^n$ (which contains $Q$) as a

hyperplane $(\mathbb{R}^n, 1) \subset \mathbb{R}^{n+1}$. Let $\sigma \subset \mathbb{R}^{n+1}$ be the cone with vertex at the origin $\underline{o}$ which is spanned by $Q$. Then $d(\sigma) = d + 1$. Nonempty faces of $\sigma$ are in bijective correspondence with faces of $Q$ (with a shift of dimension by 1), where the origin $\underline{o}$ corresponds to the empty face $\emptyset \subset Q$. Assuming the Hard Lefschetz Conjecture the polynomials $ih(\sigma)$ and $ip(\sigma)$ are defined, and by induction on dimension one concludes that

$$ih(\sigma)(q) = h(Q)(q^2), \quad ip(\sigma)(q) = g(Q)(q^2).$$

Thus Stanley's conjecture follows from the corresponding statement about the coefficients of $ih(\sigma)$.

## 8. Calai conjecture (after T. Braden and R. MacPherson)

The statement of following theorem is the $ip$-analog of the inequalities conjectured by G. Kalai and proven, in the rational case, by T. Braden and R.D. MacPherson in [BM]. Our proof follows the same pattern as the one in [BM]. However, major simpifications result from abscence of rationality hypotheses and, consequently, any ties to geometry whatsoever.

Suppose that $\sigma$ is a cone in $V$ and let $[\sigma]$ denote as usual the corresponding "affine" fan which consists of $\sigma$ and all of it's faces. Let $\tau \leq \sigma$ be a face. We know (5.2(4)) that $\mathcal{L}_{[\sigma]}|_{[\tau]} = \mathcal{L}_{[\tau]}$. Recall the graded vector spaces (7.4)

$$IP(\sigma) = \overline{\mathcal{L}_{[\sigma],\sigma}}, \quad IP(\tau) = \overline{\mathcal{L}_{[\tau],\tau}}$$

and the corresponding Poincaré polynomials $ip(\sigma)$, $ip(\tau)$. Consider the minimal sheaf $\mathcal{L}^{\tau}_{[\sigma]} \in \mathfrak{M}(\mathcal{A}_{[\sigma]})$. Its support is $\mathrm{Star}(\tau)$ and we put

$$IP(\mathrm{Star}(\tau)) := \overline{\mathcal{L}^{\tau}_{[\sigma],\sigma}}.$$

Let $ip(\mathrm{Star}(\tau))$ denote the corresponding Poincaré polynomial.

**Theorem 8.1.** *Suppose that $\sigma$ is a cone (in $V$) and $\tau$ is a face of $\sigma$. Then, there is an inequality, coefficient by coefficient, of polynomials with non-negative coefficients*

$$ip(\sigma) \geq ip(\tau) \cdot ip(Star(\tau)) \ .$$

*Proof.* Let $\iota : \mathrm{Star}(\tau) \longrightarrow \Phi$ denote the closed embedding. Then $\iota_*\iota^{-1}\mathcal{L}_{[\sigma]} \in \mathfrak{M}(\mathcal{A}_{[\sigma]})$. Indeed, the sheaf $\iota^{-1}\mathcal{L}_{[\sigma]}$ is flabby, hence so is $\iota_*\iota^{-1}\mathcal{L}_{[\sigma]}$. Moreover, $\iota_*$ is the extension by zero, so $\iota_*\iota^{-1}\mathcal{L}_{[\sigma]}$ is locally free.

Thus by the structure Theorem 5.3 there is a direct sum decomposition

$$\iota_*\iota^{-1}\mathcal{L}_{[\sigma]} \simeq \bigoplus_{\rho\geq\tau} \mathcal{L}^{\rho}_{[\sigma]} \otimes V_\rho$$

where the multiplicities $V_\rho$ are certain graded vector spaces. Comparing the stalks at $\tau$ we find that

$$\mathcal{L}_{[\sigma],\tau} \simeq \mathcal{L}^{\tau}_{[\sigma],\tau} \otimes V_\tau .$$

Hence $V_\tau = IP(\tau)$.

On the other hand, comparing the stalks at $\sigma$ we find

$$\mathcal{L}_{[\sigma],\sigma} \simeq \mathcal{L}^{\tau}_{[\sigma],\sigma} \otimes V_\tau \oplus \bigoplus_{\rho > \tau} \mathcal{L}^{\rho}_{[\sigma],\sigma} \otimes V_\rho \ .$$

In particular

$$IP(\sigma) \simeq IP(\mathrm{Star}(\tau)) \otimes IP(\tau) \oplus \bigoplus_{\rho > \tau} \overline{\mathcal{L}^{\rho}_{[\sigma],\sigma}} \otimes V_\rho \ .$$

Numerically this amounts to the inequality

$$ip(\sigma) \geq ip(\tau) ip(\mathrm{Star}(\tau)) \ .$$

$\square$

## References


[B] M. Brion, The structure of the polytope algebra, Tôhuku Math. J. **49** (1997), 1–32.

[BBD] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux Perverse, Asterisque Vol. 100, 1982.

[BL] J. Bernstein, V. Lunts, Equivariant Sheaves and Functors, LNM 1578, Springer-Verlag, 1994.

[BM] T. Braden, R.D. MacPherson, Intersection homology of toric varieties and a conjecture of Kalai, preprint, 1997.

[L] V. Lunts, Equivariant sheaves on toric varieties, Compositio Math. **96** (1995), 63-83.

[S] R. Stanley, Generalized h-vectors, intersection cohomology of toric varieties and related results, in Commutative Algebra and Combinatorics, M. Nagata and H. Matsumura, eds., Adv. Stud. Pure Math. 11, Kinokunia, Tokyo, and North-Holland, Amsterdam/New York, 1987, 187–213.



I.H.E.S., 35, route de Chartres, 91440 Bures-sur-Yvette, France
*E-mail address*: bressler@ihes.fr

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
*E-mail address*: vlunts@indiana.edu